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Digital Signal Processing

Signals & Transforms – Formula Collection

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Collection 1

Basic formulae

Trigonometric functions

$$e^{jx} = \cos x + j \sin x \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2} \qquad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin^2(x) + \cos^2(x) = 1 \qquad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} \qquad \sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \qquad \cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Hyperbolic functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \operatorname{arccsch}(x) = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$$

$$\operatorname{arccosh}(x) = \ln\left(x \pm \sqrt{x^2 - 1}\right) \qquad \operatorname{arcsech}(x) = \ln\left(\frac{1 \pm \sqrt{1 - x^2}}{x}\right)$$

$$\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \operatorname{arccoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \tanh^2(x) + \operatorname{sech}^2(x) = 1 \qquad \coth^2(x) - \operatorname{csch}^2(x) = 1$$

Signal characteristics

$$x_{PTP} = \max_{0 \le t \le T} x(t) - \min_{0 \le t \le T} x(t) \qquad x_{PTP} = \max_{0 \le n < N} x[n] - \min_{0 \le n < N} x[n]$$

$$x_{MEAN} = \frac{1}{T}\int_0^T x(t)\,dt \qquad x_{MEAN} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

$$\mu_x = \int_{-\infty}^{+\infty} x f_x(x)\,dx \qquad \mu_x = \sum_{i=-\infty}^{+\infty} x_i f_x(x_i)$$

$$x_{VAR}^2 = \frac{1}{T}\int_0^T (x(t) - x_{MEAN})^2\,\mathrm{d}t \qquad x_{VAR}^2 = \frac{1}{N-1}\sum_{n=0}^{N-1}(x[n] - x_{MEAN})^2$$

$$\sigma_x^2 = \int_{-\infty}^{+\infty}(x-\mu_x)^2 f_x(x)\,\mathrm{d}x \qquad \sigma_x^2 = \sum_{i=-\infty}^{+\infty}(x_i-\mu_x)^2 f_x(x_i)$$

$$\begin{aligned} x_{RMS} &= \sqrt{\frac{1}{T}\int_0^T x^2(t)\,\mathrm{d}t} \\ &= \sqrt{x_{MEAN}^2 + x_{VAR}^2} \end{aligned} \qquad \begin{aligned} x_{RMS} &= \sqrt{\frac{1}{N-1}\sum_0^{N-1} x^2[n]} \\ &= \sqrt{x_{MEAN}^2 + x_{VAR}^2} \end{aligned}$$

1.1 Decomposition

Interlaced decomposition

$$f[n] = f_{2n}[n] + f_{2n+1}[n]$$

with

$$f_{2n}[n] = \begin{cases} f[n] & \text{for n even} \\ 0 & \text{for n odd} \end{cases} \qquad f_{2n+1}[n] = \begin{cases} 0 & \text{for n even} \\ f[n] & \text{for n odd} \end{cases}$$

Even/Odd decomposition

$$f(t) = f_e(t) + f_o(t) \quad \text{with} \quad \begin{aligned} f_e(t) &= \frac{f(t)+f(-t)}{2} \\ f_o(t) &= \frac{f(t)-f(-t)}{2} \end{aligned}$$

$$f[n] = f_e[n] + f_o[n] \quad \text{with} \quad \begin{aligned} f_e[n] &= \frac{f[n]+f[-n]}{2} \\ f_o[n] &= \frac{f[n]-f[-n]}{2} \end{aligned}$$

Collection 2

The Fourier transform

Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, \mathrm{e}^{j\omega_k t} \qquad C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, \mathrm{e}^{-j\omega_k t}\, \mathrm{d}t \qquad \omega_k = k\frac{2\pi}{T}$$

Trigonometric Fourier Series

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{+\infty} A_k \cos\left(\omega_k t\right) + \sum_{k=1}^{+\infty} B_k \sin\left(\omega_k t\right)$$

$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(\omega_k t\right) \mathrm{d}t$$

$$B_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\omega_k t\right) \mathrm{d}t$$

$$\omega_k = k\frac{2\pi}{T}$$

Exponential to trigonometric

$$A_k = C_k + C_{-k} \qquad k \in \mathbb{Z}^+$$
$$B_k = j(C_k - C_{-k}) \qquad k \in \mathbb{Z}_0^+$$

$$C_0 = \frac{A_0}{2}$$
$$C_k = \frac{A_k - jB_k}{2} \qquad k \in \mathbb{Z}_0^+$$
$$C_{-k} = \frac{A_k + jB_k}{2} \qquad k \in \mathbb{Z}_0^+$$

Trigonometric to exponential

Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_a(\omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega \qquad X_a(\omega) = \int_{-\infty}^{+\infty} x(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t$$

Properties

Given:

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$
$$y(t) \xrightarrow{\mathcal{F}} Y(\omega)$$
$$a, b \in \mathbb{R}$$
$$k \in \mathbb{R}_0$$

Linearity

$$ax(t) + by(t) \xrightarrow{\mathcal{F}} aX(\omega) + bY(\omega)$$

Time-frequency symmetry

$$X(t) \xrightarrow{\mathcal{F}} 2\pi x(-\omega)$$

Time/Frequency scaling

$$x(kt) \xrightarrow{\mathcal{F}} \frac{1}{|k|} X\left(\frac{\omega}{k}\right)$$
$$\frac{1}{|k|} x\left(\frac{t}{k}\right) \xrightarrow{\mathcal{F}} X(k\omega)$$

Time/Frequency shifting

$$x(t-t_0) \xrightarrow{\mathcal{F}} X(\omega)\,\mathrm{e}^{-j\omega t_0}$$

$$x(t)\,\mathrm{e}^{+j\omega_0 t} \xrightarrow{\mathcal{F}} X(\omega-\omega_0)$$

Time/Frequency differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \xrightarrow{\mathcal{F}} j\omega X(\omega)$$

$$-jtx(t) \xrightarrow{\mathcal{F}} \frac{\mathrm{d}}{\mathrm{d}\omega}X(\omega)$$

Parseval's theorem

$$\int_{-\infty}^{+\infty} x(t)y^*(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)Y^*(\omega)\,\mathrm{d}\omega$$

$$\int_{-\infty}^{+\infty} \left|x(t)\right|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|X(\omega)\right|^2\,\mathrm{d}\omega$$

Convolution/Multiplication

$$x(t)\star y(t) \xrightarrow{\mathcal{F}} X(\omega)Y(\omega)$$

$$x(t)y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}X(\omega)\star Y(\omega)$$

Discrete-time Fourier Transform (DtFT)

$$X_p(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\,\mathrm{e}^{-j\omega nT_s} \qquad x[n] = \frac{1}{\omega_s}\int_{-\frac{\omega_s}{2}}^{+\frac{\omega_s}{2}} X_p(\omega)\,\mathrm{e}^{j\omega nT_s}\,\mathrm{d}\omega$$

Shannon's Time-sampling Theorem To be able to sample and reconstruct a signal with a maximum frequency content of $\pm\omega_B$ one must ensure that the sampling frequency ω_s fulfills:

$$\omega_s \geq 2\omega_B$$

Discrete Fourier Transform

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_p[k]\,\mathrm{e}^{j\frac{2\pi kn}{N}} \qquad X_p[k] = \sum_{n=0}^{N-1} x_p[n]\,\mathrm{e}^{-j\frac{2\pi kn}{N}}$$

Properties of the DFT

Given:

$$x_p[n] \xrightarrow{\text{DFT}} X_p[k]$$

$$y_p[n] \xrightarrow{\text{DFT}} Y_p[k]$$

$$a, b \in \mathbb{R}$$

$$u \in \mathbb{R}_0$$

Linearity

$$ax_p[n] + by_p[n] \xrightarrow{\text{DFT}} aX_p[k] + bY_p[k]$$

Time-frequency symmetry

$$X_p[n] \xrightarrow{\text{DFT}} Nx_p[-k]$$

Cyclic Time/Frequency shifting

$$x_p[n-i] \xrightarrow{\text{DFT}} X_p[k]\,\mathrm{e}^{-j\frac{2\pi ki}{N}}$$

$$x_p[n]\,\mathrm{e}^{+j\frac{2\pi in}{N}} \xrightarrow{\text{DFT}} X_p[k-i]$$

Parseval's theorem

$$\sum_{n=0}^{N-1} x_p[n]y_p^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_p[k]Y_p^*[k]$$

$$\sum_{n=0}^{N-1} \left|x_p[n]\right|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \left|X_p[k]\right|^2$$

Convolution/Multiplication

$$x[n]\star y[n] \xrightarrow{\text{DFT}} X[k]Y[k]$$

$$x[n]\times y[n] \xrightarrow{\text{DFT}} \frac{1}{N}X[k]\star Y[k]$$

Conversion Diagram

$$\begin{tikzcd}
X_a[k] \arrow[rrr, "2\pi\text{Ш}_{\Omega}(\omega)"] & & & X_a(\omega) \\
& x_p(t) \arrow[ul, "\text{Fourier Series}" description] \arrow[r, "1"] & x_a(t) \arrow[ur, "\text{Fourier Transform}" description] & \\
& x_p[n] \arrow[ur, "\cancel{T}_s\text{Ш}_{T_s}(t)"] \arrow[r, "1"'] \arrow[dl, "\text{Discrete Fourier Transform}" description] & x_a[n] \arrow[u, "\cancel{T}_s\text{Ш}_{T_s}(t)"'] \arrow[dr, "\text{Discrete-time Fourier Transform}" description] & \\
X_p[k] \arrow[rrr, "\omega_f\text{Ш}_{\omega_f}(\omega)"'] \arrow[uuurrr, bend right=60, "\frac{2\pi}{N}\text{Ш}_{\omega_f}(\omega)"'] & & & X_p(\omega) \arrow[uuu, "T_s"']
\end{tikzcd}$$

Analysis windows

Rectangular window

$$r_N[n] = \begin{cases} 1 & \text{if } 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

Bartlett window

$$t_N[n] = \begin{cases} \frac{n}{N/2} & \text{if } 0 \leq n \leq N/2 \\ 1 - \frac{n-N/2}{N/2} & \text{if } N/2 < n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Hann window

$$\text{hn}_N[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Hamming window

$$\text{hm}_N[n] = \begin{cases} 0.53836 - 0.46164\cos\left(\frac{2\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Blackman window

$$\text{bm}_N[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Kaiser windows

$$\text{k}_N[n] = \begin{cases} \dfrac{I_0\left(\pi\alpha\sqrt{1-\left(\frac{2n}{N}-1\right)^2}\right)}{I_0(\pi\alpha)} & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

with I_0 the zeroth-order modified Bessel function of the first kind:

$$I_0(x) = 1 + \sum_{k=1}^{+\infty} \frac{x^{2k}}{2^{2k}(k!)^2}$$

Nuttall window

$$\mathrm{na}_N[n] = \begin{cases} a_0 - a_1 \cos\left(\frac{2\pi n}{N}\right) + a_2 \cos\left(\frac{4\pi n}{N}\right) - a_3 \cos\left(\frac{6\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

with

$$a_0 = 0.355768 \qquad a_1 = 0.487396 \qquad a_2 = 0.144232 \qquad a_3 = 0.012604$$

Blackman-Harris window

$$\mathrm{bh}_N[n] = \begin{cases} a_0 - a_1 \cos\left(\frac{2\pi n}{N}\right) + a_2 \cos\left(\frac{4\pi n}{N}\right) - a_3 \cos\left(\frac{6\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

with

$$a_0 = 0.35875 \qquad a_1 = 0.48829 \qquad a_2 = 0.14128 \qquad a_3 = 0.01168$$

Blackman-Nuttall window

$$\mathrm{bna}_N[n] = \begin{cases} a_0 - a_1 \cos\left(\frac{2\pi n}{N}\right) + a_2 \cos\left(\frac{4\pi n}{N}\right) - a_3 \cos\left(\frac{6\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

with

$$a_0 = 0.3635819 \qquad a_1 = 0.4891775 \qquad a_2 = 0.1365995 \qquad a_3 = 0.0106411$$

Flat top window

$$\mathrm{ft}_N[n] = \begin{cases} a_0 - a_1 \cos\left(\frac{2\pi n}{N}\right) + a_2 \cos\left(\frac{4\pi n}{N}\right) - a_3 \cos\left(\frac{6\pi n}{N}\right) + a_4 \cos\left(\frac{8\pi n}{N}\right) & \text{if } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

with

$$a_0 = 1 \qquad a_1 = 1.93 \qquad a_2 = 1.29 \qquad a_3 = 0.388 \qquad a_4 = 0.032$$

Collection 3

Sampling, Quantization and Reconstruction

Low-pass sampling

$$\omega_s \geq 2\omega_B.$$

Band-pass sampling

$$\underbrace{\frac{2\omega_0 + \omega_B}{n+1}}_{\omega_{s,nL}} \leq \omega_s \leq \underbrace{\frac{2\omega_0 - \omega_B}{n}}_{\omega_{s,nU}}$$

Reconstruction

$$zoh(t) = \begin{cases} 1/T_s & \text{if } t \in [0, T_s] \\ 0 & \text{if } t \notin [0, T_s] \end{cases}$$

$$ZOH(\omega) = \operatorname{sinc}\left(\frac{\omega T_s}{2}\right) e^{-j\frac{\omega T_s}{2}}$$

Quantization

N-bit mid-rise quantization

$$\hat{x} = \frac{0.5 + \left\lfloor \frac{x}{\Delta} \right\rfloor}{2^{N-1}} x_{\max} \qquad \text{with} \qquad \Delta = \frac{2x_{\max}}{2^N}$$

N-bit mid-tread quantization

$$\hat{x} = \frac{\left\lfloor \frac{x}{\Delta} + 0.5 \right\rfloor}{2^{N-1} - 0.5} x_{\max} \qquad \text{with} \qquad \Delta = \frac{2x_{\max}}{2^N - 1}$$

Uniform quantization error model

$$\sigma(\varepsilon) = \frac{\Delta}{\sqrt{12}} \qquad \mathrm{PSD}_n(\omega) = \frac{\Delta^2}{12\omega_s} \qquad \mathrm{PSM}_n[k] = \frac{\Delta^2}{12N}$$

ADC and DAC Static performance

Offset

$$E_{offset,\mathrm{ADC}} = \frac{\sum_k (t_k - k\Delta)}{2^N - 1}$$

$$E_{offset,\mathrm{DAC}} = \frac{\sum_k (r_k - (k + 0.5)\Delta)}{2^N}$$

Full-scale gain error

$$E_{FSG,\mathrm{ADC}} = t_{2^{N-1}-1} - t_{-2^{N-1}+1} - (2^N - 2)\Delta$$

$$E_{FSG,\mathrm{DAC}} = r_{2^{N-1}-1} - r_{-2^{N-1}} - (2^N - 1)\Delta$$

Differential nonlinearity (DNL)

$$\mathrm{DNL}_{\mathrm{ADC}} = \frac{\max_k \left| t_k - t_{k-1} - \Delta \right|}{\Delta}$$

$$\mathrm{DNL}_{\mathrm{DAC}} = \frac{\max_k \left| r_k - r_{k-1} - \Delta \right|}{\Delta}$$

Integral nonlinearity (INL)

$$\mathrm{INL}_{\mathrm{ADC}} = \frac{\max_k (t_k - k\Delta) - \min_k (t_k - k\Delta)}{\Delta}$$

$$\mathrm{INL}_{\mathrm{DAC}} = \frac{\max_k (r_k - (k + 0.5)\Delta) - \min_k (r_k - (k + 0.5)\Delta)}{\Delta}$$

ADC and DAC Dynamic performance

Signal-to-noise ratio (S/N)

$$\mathrm{S/N} = \frac{V^2_{\mathrm{RMS},signal}}{V^2_{\mathrm{RMS},noise}}$$

Total harmonic distortion (THD)

$$\mathrm{THD} = \frac{\sum_{i=2}^{+\infty} V^2_{\mathrm{RMS},i}}{V^2_{\mathrm{RMS},signal}}$$

Signal-to-noise-and-distortion ratio (SNAD)

$$\mathrm{SNAD} = \frac{V^2_{\mathrm{RMS},signal}}{V^2_{\mathrm{RMS},noise} + \sum_{i=2}^{+\infty} V^2_{\mathrm{RMS},i}}$$

Spurious Free Dynamic Range

$$\mathrm{SFDR} = \frac{V^2_{signal}}{\max_{i\geq 2} V_i^2}$$

Collection 4

The Laplace transform

Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,\mathrm{e}^{st}\,\mathrm{d}s \qquad X(s) = \int_{-\infty}^{+\infty} x(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

One-sided Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,\mathrm{e}^{st}\,\mathrm{d}s \qquad X(s) = \int_{0^-}^{+\infty} x(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

Lemma: convergence of the Laplace transform

For any signal $x(t)$ that is

1. causal,
2. piecewise continuous on the range $t \in [0, t_0]$, and
3. exponentially bounded otherwise, i.e.

$$\exists t_0, M, \alpha \in \mathbb{R}, \forall t > t_0 : \left|x(t)\right| < M\,\mathrm{e}^{\alpha t},$$

the corresponding Laplace transform $X(s) = \mathcal{L}(x(t))$ converges for $\Re s > \alpha$.

Properties

Given:

$$x(t) \xrightarrow{\mathcal{L}} X(s)$$
$$y(t) \xrightarrow{\mathcal{L}} Y(s)$$
$$a, b \in \mathbb{R}$$
$$k \in \mathbb{R}_0$$

and $x(t)$ and $y(t)$ are causal.

Linearity

$$ax(t) + by(t) \xrightarrow{\mathcal{L}} aX(s) + bY(s)$$

Time/Frequency scaling

$$x(kt) \xrightarrow{\mathcal{L}} \frac{1}{|k|}X\left(\frac{s}{k}\right)$$
$$\frac{1}{|k|}x\left(\frac{t}{k}\right) \xrightarrow{\mathcal{L}} X(ks)$$

Time/Frequency shifting

$$x(t-t_0) \xrightarrow{\mathcal{L}} X(s)\,\mathrm{e}^{-st_0}$$
$$x(t)\,\mathrm{e}^{+s_0 t} \xrightarrow{\mathcal{L}} X(s-s_0)$$

Time/Frequency differentation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \xrightarrow{\mathcal{L}} sX(s) - x(0)$$
$$-tx(t) \xrightarrow{\mathcal{L}} \frac{\mathrm{d}}{\mathrm{d}s}X(s)$$

Time/Freqency division

$$\frac{x(t)}{t} \xrightarrow{\mathcal{L}} \int_s^{\infty} X(u)\,\mathrm{d}u$$
$$\int_0^t x(u)\,\mathrm{d}u \xrightarrow{\mathcal{L}} \frac{X(s)}{s}$$

Convolution/Multiplication

$$x(t) \star y(t) \xrightarrow{\mathcal{L}} X(s)Y(s)$$

$$x(t)y(t) \xrightarrow{\mathcal{L}} \frac{1}{2\pi j} X(s) \star Y(s)$$

Initial/Final value theorem

$$x(0) = \lim_{s \to \infty} sX(s)$$

$$\lim_{t \to +\infty} x(t) = \lim_{s \to 0} sX(s)$$

Laplace transform of a periodical function

$$\left. \begin{array}{r} f(t) \text{ is periodical with period } T \\ \phi(t) = (u(t) - u(t-T))f(t) \xrightarrow{\mathcal{L}} \Phi(s) \end{array} \right\} \Rightarrow f(t) \xrightarrow{\mathcal{L}} F(s) = \frac{\Phi(s)}{1 - \mathrm{e}^{-sT}}$$

Common transform pairs

$x(t)$	$\xrightarrow{\mathcal{L}}$	$X(s)$
$\delta(t)$	$\xrightarrow{\mathcal{L}}$	1
$u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{1}{s}$
$t \cdot u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{1}{s^2}$
$\frac{t^n}{n!} \cdot u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{1}{s^{n+1}}$
$t^p \cdot u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\mathrm{e}^{-at} \cdot u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{1}{s+a}$
$\sin(at) \cdot u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{a}{s^2+a^2}$
$\cos(at) \cdot u(t)$	$\xrightarrow{\mathcal{L}}$	$\frac{s}{s^2+a^2}$

Collection 5

The Z-transform

Z-transform

Given a time-domain signal $x[n]$, we can calculate a frequency domain representation $X(z)$, the Z-transform of the signal, from which the original time-domain signal can be recovered by inverse transformation:

$$x[n] = \frac{1}{2\pi j}\int_{e^{\sigma T_s - j\pi}}^{e^{\sigma T_s + j\pi}} X(z) z^{n-1} dz \qquad X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

One-sided Z-transform

Given a causal time-domain signal $x[n]$, we can calculate a frequency domain representation $X(z)$, the Z-transform of the signal, from which the original time-domain signal can be recovered by inverse transformation:

$$x[n] = \frac{1}{2\pi j}\int_{e^{\sigma T_s - j\pi}}^{e^{\sigma T_s + j\pi}} X(z) z^{n-1} dz \qquad X(z) = \sum_{n=0}^{+\infty} x[n] z^{-n}$$

Lemma: convergence of the Z-transform

For any signal $x[n]$ that is

1. causal,
2. bounded for $n \in [0, N]$, and
3. exponentially bounded otherwise, i.e.

$$\exists M \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n > N : \left|x[n]\right| < M^n,$$

the corresponding Z-transform $X(z) = \mathcal{Z}(x[n])$ converges for $|z| > M$.

Properties

Given:

$$x[n] \xrightarrow{\mathcal{Z}} X(z)$$
$$y[n] \xrightarrow{\mathcal{Z}} Y(z)$$
$$a, b \in \mathbb{R}$$
$$n_0 \in \mathbb{N}_0$$

Linearity

$$ax[n] + by[n] \xrightarrow{\mathcal{Z}} aX(z) + bY(z)$$

Z-scaling

$$a^n x[n] \xrightarrow{\mathcal{Z}} X\left(\frac{z}{a}\right)$$

Time shifting

$$x[n - n_0] \xrightarrow{\mathcal{Z}} z^{-n_0} X(z)$$
$$x[n + n_0] \xrightarrow{\mathcal{Z}} z^{n_0}\left(X(z) - \sum_{i=0}^{n_0 - 1} x[i] z^{-i}\right)$$

Time reversal

$$x[-n] \xrightarrow{\mathcal{Z}} X(z^{-1})$$

Time difference

$$x[n] - x[n - n_0] \xrightarrow{\mathcal{Z}} \frac{z^{n_0} - 1}{z^{n_0}} X(z)$$

Z-Differentiation

$$nx[n] \xrightarrow{\mathcal{Z}} -z\frac{dX(z)}{dz}$$

Summation

$$\sum_{i=0}^{n_0} x[n-i] \xrightarrow{\mathcal{Z}} \frac{z - z^{-n_0}}{z-1} X(z)$$

$$\sum_{i=0}^{+\infty} x[n-i] \xrightarrow{\mathcal{Z}} \frac{z}{z-1} X(z)$$

Convolution

$$x[n] \star y[n] \xrightarrow{\mathcal{Z}} X(z)Y(z)$$

Initial/Final value theorem

$$x[0] = \lim_{z \to \infty} X(z)$$

$$\lim_{n \to +\infty} x[n] = \lim_{z \to 1} (z-1)\, X(z)$$

Arbitrary value theorem

$$x[n] = \lim_{z \to \infty} z^n \left(X(z) - \sum_{i=0}^{n-1} x[i] z^{-i} \right)$$

Common transform pairs

$x[n]$	$\xrightarrow{\mathcal{Z}}$	$X(z)$
$\delta[n]$	$\xrightarrow{\mathcal{Z}}$	1
$u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z}{z-1}$
$nu[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z}{(z-1)^2}$
$n^2u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z(z+1)}{(z-1)^3}$
$n^3u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
na^n	$\xrightarrow{\mathcal{Z}}$	$\frac{az}{(z-a)^2}$
$\mathrm{e}^{an}u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z}{z-\mathrm{e}^a}$
$\frac{a^n}{n!}$	$\xrightarrow{\mathcal{Z}}$	$\mathrm{e}^{\frac{a}{z}}$
$\sin(an)u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z \sin a}{z^2 - 2z\cos a + 1}$
$\cos(an)u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z(z - \cos a)}{z^2 - 2z\cos a + 1}$
$\sinh(an)u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z \sinh a}{z^2 - 2z\cosh a + 1}$
$\cosh(an)u[n]$	$\xrightarrow{\mathcal{Z}}$	$\frac{z(z - \cosh a)}{z^2 - 2z\cosh a + 1}$