Academic year

2025-2026

Faculty of Applied Engineering

Systems Theory

Solutions to the Exercises

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1512FTISYS 5-Systems Theory

University of Antwerp

This document has been typeset using LaTeX and the `uantwerpendocs` package.
Calculations have been performed using Matlab/Octave and generic programming and script languages (C/C++/Perl/Python).
Graphics have been composed using PGF, TiKZ and InkScape.
All this material has been prepared on a GNU/Linux workstation.

Typesetting of this document was enabled by:

Digital Manifold Waves
Independent Publishing Cooperation

DSP-ST-2025-1.00-TB

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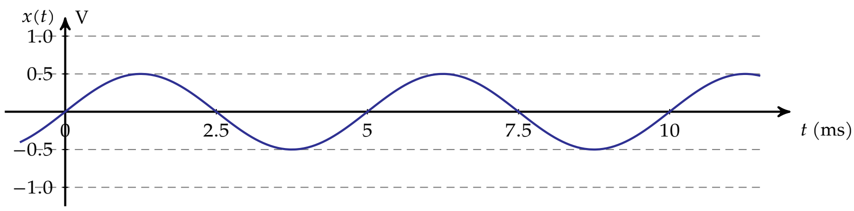
Chapter 2
Signals

Solution 2.4.1-1:
The drawing of the sine wave $x(t)$ can be found below:



Its mathematical expression is:

$$x(t) = 0.5 \cdot \sin(2\pi \cdot 200 \cdot t) \qquad (\text{V})$$

Note that though we wrote the unit at the end of the expression above, we did not write the unit Hz inside the sine wave. We rarely write units inside the argument of a sine wave. However, they are silently there, i.e. we assume the time t is its value in seconds.

The period and the pulsation can be calculated to be:

$$T = \frac{1}{f} = 5\,\text{ms} \qquad\qquad \omega = 2\pi f = 2\pi \cdot 200\,\text{Hz} = 1\,256.6\,\text{rad/s}$$

Solution 2.4.1-2:
A possible solution is:

$$f(t) = 0.75\,\text{V} \cdot \cos\left(\frac{2\pi}{3 \times 10^{-9}} t - \frac{7\pi}{6}\right) = 0.75\,\text{V} \cdot \cos\left(2.094\,4 \times 10^{9} \cdot t - \frac{7\pi}{6}\right)$$

assuming t is the value in unit s. We typically do not write these units inside the argument of a sine wave. However, we silently imply they are there!
We could also use a sine wave to describ the same waveform:

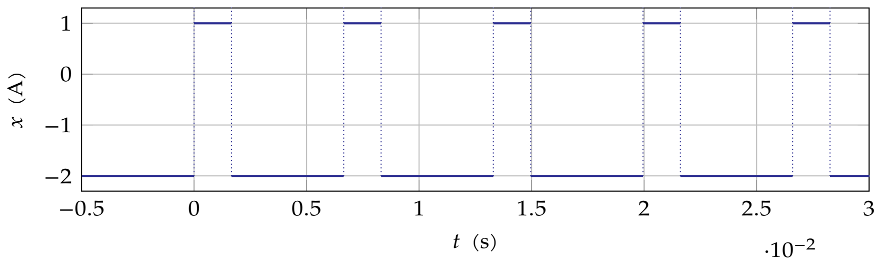
$$f(t) = 0.75\,\text{V} \cdot \sin\left(\frac{2\pi}{3 \times 10^{-9}} t - \frac{\pi}{3}\right) = 0.75\,\text{V} \cdot \cos\left(2.094\,4 \times 10^{9} \cdot t - \frac{\pi}{3}\right)$$

Solution 2.4.1-3:
The expression is:

$$x(t) = 0.5\,\mu\text{V} \cdot \sin\left(\frac{2\pi}{3 \times 10^{-3}}(t - 0.5 \times 10^{-3})\right) = 0.5\,\mu\text{V} \cdot \sin(2094.4 \cdot t - 1.0472)$$

in which we assume the time t to be in seconds.

Solution 2.4.1-4:
This is a graph of the square wave:



Solution 2.4.1-5:
The frequency f and symmetry s can be calculated to be:

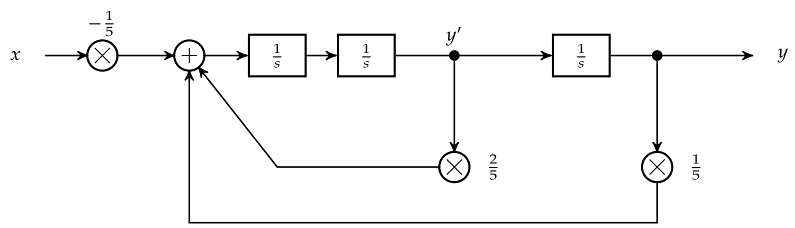
$$f = \frac{1}{T} = \frac{1}{3\,\text{ns}} = 333.333\,\text{MHz}$$
$$s = \frac{1\,\text{ns}}{3\,\text{ns}} = 0.333\,\%$$

Chapter 3
Systems

Solution 3.3.4-1:
The first step is to rewrite this equation, such that the highest derivative of y is solved from this equation:

$$\frac{d^3y}{dt^3} = \frac{2}{5}\frac{dy}{dt} + \frac{1}{5}y - \frac{1}{5}x$$

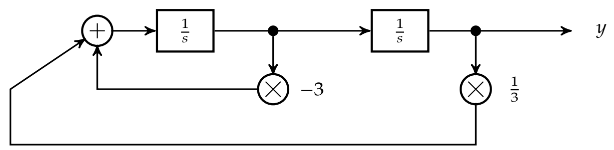
This results in the following block diagram:



Solution 3.3.4-2:
The first step is to rewrite this equation, such that the highest derivative of y is solved from this equation:

$$\frac{d^2y}{dt^2} = -3\frac{dy}{dt} + \frac{1}{3}y$$

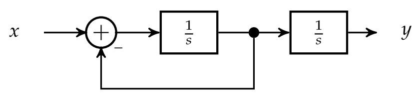
This results in the following block diagram:



Solution 3.3.4-3:
The first step is to rewrite this equation, such that the highest derivative of y is solved from this equation:

$$\frac{d^2y}{dt^2} = -\frac{dy}{dt} + x$$

This results in the following block diagram:



Solution 3.3.4-4:
The signal at the input of the leftmost integrator is y''. Therefore:

$$y'' = -4y' - \frac{3}{2}y + \pi x$$
$$\Leftrightarrow \quad y'' + 4y' + \frac{3}{2}y = \pi x$$

Solution 3.3.4-5:
The signal at the input of the leftmost integrator is y'''. Therefore:

$$y''' = 3y'' - 6y' + \pi x$$
$$\Leftrightarrow \quad y''' - 3y'' + 6y' = \pi x$$

Solution 3.3.4-6:
The signal at the input of the rightmost integrator is y''. Therefore:

$$y'' = 5y' - 7y$$
$$\Leftrightarrow \quad y'' - 5y' + 7y = 0$$

Solution 3.3.5.2-1:
We know that the impulse response can be calculated as the derivative of the step response multiplied by the unit second:

$$h(t) = \mathrm{s} \cdot \frac{\mathrm{d} \sin(20t)}{\mathrm{d}t}$$

We need to remember that the coefficient 20 in the argument of the sine wave actually means 20 radians per second! This explains the following:

$$h(t) = 20 \cos(20t)$$

Solution 3.3.5.2-2:

Direct method
We know from our early lectures on analog electronics, that this circuit is a potentiometric divider, therefore:

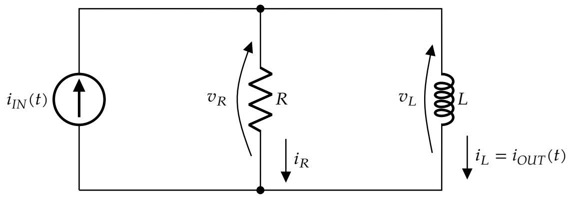
$$v_{OUT}(t) = \frac{R_2}{R_1 + R_2} v_{IN}(t)$$
$$\Big\downarrow \, v_{IN}(t) = \delta(t)$$
$$h(t) = \frac{R_2}{R_1 + R_2}\,\delta(t)$$

Indirect method
We first determine the step response. Given the potentiometric divider, we obtain:

$$v_{OUT}(t) = \frac{R_2}{R_1 + R_2}\,\mathrm{u}(t)$$
$$\Big\downarrow \, h(t) = \mathrm{s} \cdot \frac{\mathrm{d}v_{OUT}(t)}{\mathrm{d}t}$$
$$h(t) = \mathrm{s} \cdot \frac{R_2}{R_1 + R_2}\,\delta(t) \cdot \frac{1}{\mathrm{s}} = \frac{R_2}{R_1 + R_2}\,\delta(t)$$

The unit 1 over second in the bottom line is caused by applying the chain rule during the derivation of $\mathrm{u}(1\,\mathrm{Hz} \cdot t)$ (which is the explicit version of $\mathrm{u}(t)$).

Solution 3.3.5.2-3:
Let's first indicate some extra variables on the schematic:



We willuse these variables below.

Direct method
Let's consider the three phases of the Dirac impulse.

- $t < 0 \Rightarrow i_{OUT}(t) = 0$ (as we assume that any energy in the coil will have dissipated itself in the resistor R in the infinite amount of time until $t = 0$)
- $t = 0$:
  The current spike is infinitely high. As the coil resists conducting current, we may assume that the amount of current that the coil experiences from the spike is finite. Therefore 'the infiniteness of the current' fully goes through the resistor, leading to a voltage over the resistor equal to:
  $$v_R(0) = 1\,\text{k}\Omega \cdot \delta(t) = \delta(0)\ \text{kV}$$
  This voltage spike will be integrated by the coil, based on the coil's branch equation:
  $$\begin{aligned} v_L &= L\frac{\mathrm{d}i_L}{\mathrm{d}t} \\ \Leftrightarrow \quad i_L(0) &= \frac{1}{L}\int_{0^-}^{0^+} v_R(t)\,\mathrm{d}t \\ \Leftrightarrow \quad i_L(0) &= \frac{1}{10^{-3}\,\text{V s/A}}\int_{0^-}^{0^+} \delta(t)\ \text{kV}\,\mathrm{d}t = 1\,\text{MA} \end{aligned}$$
  This is the initial condition for the next phase.
- $t > 0 \Rightarrow i_{OUT}(t) = 0$:
  Assuming $v_L = v_R$ (KVL) with the voltage arrows pointing upward, and $i_R = -i_L$ (KCL) withe current arrows pointing downward, we can write the following (using the branch equations):
  $$v_L = L\frac{\mathrm{d}i_L}{\mathrm{d}t}$$
  $$v_L = v_R = Ri_R(t) = -Ri_L$$
  Eliminating v_L and keeping i_{OUT}, this leads to:
  $$L\frac{\mathrm{d}i_L}{\mathrm{d}t} + Ri_L = 0$$
  Solving this equation subject to the initial condition we calculated earlier, leads to:
  $$i_L(t) = 1\,\text{MA} \cdot \mathrm{e}^{-\frac{t}{\tau}}$$
  with $\tau = L/R = 1\,\mu\text{s}$.

Indirect method
Let's first calculate the step response, and therefore we consider the two phases of the step

- $t < 0 \Rightarrow i_{OUT}(t) = 0$ (as we assume that any energy in the coil will have dissipated itself in the resistor R in the infinite amount of time until $t = 0$)

- $t \geq 0$:
  As the current through an inductor cannot change abruptly (under normal circumstances), we may assume $i_L(0) = 0$. KCL on the top node, allows us to write:

  $$i_{IN} = i_R + i_L$$

  Introducing the branch equations, keeping i_L, leads to:

  $$i_{IN} = \frac{L}{R}\frac{\mathrm{d}i_L}{\mathrm{d}t} + i_L$$
  $$\Leftrightarrow \quad \tau\frac{\mathrm{d}i_L}{\mathrm{d}t} + i_L = i_{IN}$$

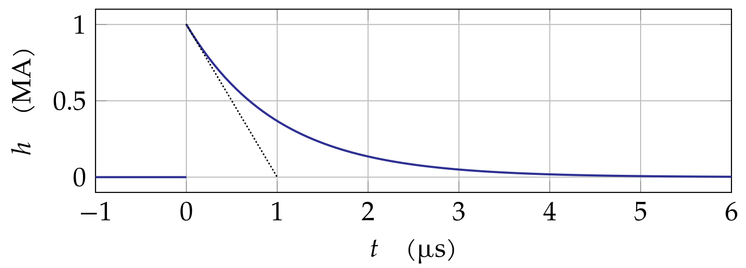
  We know that for $t > 0$, $i_{IN} = 1$, with the initial condition $i_L(0) = 0$. Solving this equation subject to this initial condition, leads to:

  $$i_L(t) = (1 - \mathrm{e}^{-\frac{t}{\tau}}) \cdot \mathrm{A}$$

  with $\tau = L/R$ Deriving and multiplying with the unit second, yields us the impulse response:

  $$h(t) = \mathrm{s} \cdot \frac{\mathrm{d}i_L}{\mathrm{d}t} = \frac{\mathrm{s}}{\tau}\mathrm{e}^{-\frac{t}{\tau}} \cdot \mathrm{A} = 1\,\mathrm{MA} \cdot \mathrm{e}^{-\frac{t}{\tau}}$$

Both methods lead to the following graph of the impulse response $h(t)$:



Chapter 4
Convolution

Solution 4.4-1:

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - \mathrm{e}^{-\frac{t}{RC}} & \text{if } 0 \leq t \leq 1 \\ \mathrm{e}^{-\frac{t}{RC}}\left(\mathrm{e}^{\frac{1}{RC}} - 1\right) & \text{if } 1 < t \end{cases}$$

Solution 4.4-2:

$$y(t) = \frac{3}{1 + (\omega RC)^2}\left(\sin \omega t - \omega RC \cos \omega t + \omega RC\, \mathrm{e}^{-\frac{t}{RC}}\right)$$

Solution 4.4-3:

$$y(t) = -\frac{6}{RC}\, \mathrm{e}^{-\frac{t}{RC}}$$

Solution 4.4-4:
Below, you can find the proof:

Commutativity of the convolution operation
Consider two signals f and g. For these signals we can elaborate the following:

$$\begin{aligned} (f \star g)(t) &= \int_{-\infty}^{+\infty} f(u) \cdot g(t-u)\, \mathrm{d}u \\ &\downarrow \text{perform the following substitution: } w = t - u \Leftrightarrow u = t - w \Leftrightarrow \mathrm{d}u = -\mathrm{d}w \\ &= \int_{+\infty}^{-\infty} f(t-w) g(w) (-\mathrm{d}w) \\ &= \int_{-\infty}^{+\infty} g(w) f(t-w)\, \mathrm{d}w \\ &= (g \star f)(t) \end{aligned}$$

■

Solution 4.4-5:
Below, you can find the proof:

Associativity of the convolution operation

Consider three signals f, g and h. For these signals we can elaborate the following:

$$\begin{aligned}
((f\star g)\star h)(t) &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(w)g(u-w)\,\mathrm{d}w\right)h(t-u)\,\mathrm{d}u\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(w)g(u-w)h(t-u)\,\mathrm{d}w\,\mathrm{d}u\\
&\downarrow \text{change the order of integration}\\
&= \int_{-\infty}^{+\infty} f(w)\int_{-\infty}^{+\infty} g(u-w)h(t-u)\,\mathrm{d}u\,\mathrm{d}w\\
&\downarrow \text{substitute } \alpha = u-w \text{ in the inner integral}\\
&= \int_{-\infty}^{+\infty} f(w)\underbrace{\int_{-\infty}^{+\infty} g(\alpha)h(t-\alpha-w)\,\mathrm{d}\alpha}_{=(g\star h)(t-w)}\,\mathrm{d}w\\
&= \int_{-\infty}^{+\infty} f(w)(g\star h)(t-w)\,\mathrm{d}w\\
&= (f\star(g\star h))(t)
\end{aligned}$$

■

Solution 4.4-6:
Below, you can find the proof:

The Dirac impulse $\delta(t)$ is the neutral element for the convolution operation
Consider a signal f. For this signal we can elaborate the following:

$$\begin{aligned}
(f\star\delta())\,(t) &= \int_{-\infty}^{+\infty} f(u)\delta(t-u)\,\mathrm{d}u\\
&\downarrow \text{sifting property of the Dirac impulse}\\
&= f(t)
\end{aligned}$$

Given the fact that the convolution operation is also commutative, the Dirac impulse $\delta(t)$ is a true neutral element for the operation. ■

Solution 4.4-7:
Below, you can find the proof:

Distributivity of the convolution operation
Consider three signals f, g and h. For these signals we can elaborate the following:

$$\begin{aligned}
(f\star(g+h))(t) &= \int_{-\infty}^{+\infty} f(u)(g+h)(t-u)\,\mathrm{d}u\\
&= \int_{-\infty}^{+\infty} (f(u)g(t-u)+f(u)h(t-u))\,\mathrm{d}u\\
&= \int_{-\infty}^{+\infty} f(u)g(t-u)\,\mathrm{d}u + \int_{-\infty}^{+\infty} f(u)h(t-u)\,\mathrm{d}u\\
&= (f\star g)(t) + (f\star h)(t)
\end{aligned}$$

■

Solution 4.4-8:
Below, you can find the proof:

Simplification of the convolution integral for causal signals
Consider two causal signals f and g, i.e. $f(t<0)=0$ and $g(t<0)=0$. The convolution integral can be split into parts:

$$\begin{aligned}
(f\star g)(t) &= \int_{-\infty}^{+\infty} f(u)g(t-u)\,\mathrm{d}u\\
&= \int_{-\infty}^{0} f(u)g(t-u)\,\mathrm{d}u + \int_{0}^{t} f(u)g(t-u)\,\mathrm{d}u + \int_{t}^{+\infty} f(u)g(t-u)\,\mathrm{d}u
\end{aligned}$$

As $u<0$ results in $f(u)=0$, the integrand of the first integral is zero over its integration range. As $t-u<0 \Leftrightarrow t<u$ results in $g(t-u)<0$, the integrand of the third integral is zero over its integration range.

Therefore, the first and third integral amount to zero. Conclusion of this all:

$$(f \star g)(t) = \int_0^t f(u)g(t-u)\,\mathrm{d}u$$

■

Chapter 5

The Continous-time Fourier transform

Solution 5.2.5-1:
$k = \frac{4}{\pi} = 1.2732$

Solution 5.3.1-1:
The trigonometric Fourier series coefficients are defined for $k \in \mathbb{Z}^+$ as:

$$\begin{aligned} A_k &= \frac{2}{T}\int_{-T/2}^{+T/2} \cos(\omega_k t)\,\mathrm{d}t \\ B_k &= \frac{2}{T}\int_{-T/2}^{+T/2} \sin(\omega_k t)\,\mathrm{d}t \end{aligned}$$

with $\omega_k = 2k\pi/T$.
Because of the fact that an integral of an odd function multiplied with an even function over a symmetrical range equals zero, we must conclude:

$$A_k = 0$$

Therefore we can focus on B_k for $k \in \mathbb{Z}_0^+$.
Given

$$x(t) = \frac{3\pi}{2}t - \frac{3\pi}{2}, \qquad t \in [0, 2]$$

and changing the integration range from $[-T/2, T/2]$ to $[0, T]$, we obtain

$$\begin{aligned} B_k &= \int_0^2 \left(\frac{3\pi}{2}t - \frac{3\pi}{2}\right)\sin(\omega_k t)\,\mathrm{d}t, \qquad \text{with } \omega_k = k\pi \\ &= \frac{3\pi}{2}\left(\int_0^2 t\sin(k\pi t)\,\mathrm{d}t - \underbrace{\int_0^2 \sin(k\pi t)\,\mathrm{d}t}_{=0,\forall k\in\mathbb{Z}^+}\right) \\ &\downarrow \text{prepare for partial integration} \\ &= -\frac{3\pi}{2}\frac{1}{k\pi}\int_{t=0}^{t=2} t\,\mathrm{d}(\cos(k\pi t)) \\ &\downarrow \text{partial integration} \\ &= -\frac{3}{2k}\left([t\cos(k\pi t)]_{t=0}^{t=2} - \underbrace{\int_0^2 \cos(k\pi t)\,\mathrm{d}t}_{=0,\forall k\in\mathbb{Z}_0^+}\right) \\ &= -\frac{3}{2k}\left(2\cos(2k\pi) - 0\right) \\ &= -\frac{3}{k} \end{aligned}$$

This leads to the following conclusion:

$$A_k = 0, \qquad \forall k \in \mathbb{Z}^+$$
$$B_k = \begin{cases} 0, & k = 0 \\ \frac{-3}{k}, & k \in \mathbb{Z}_0^+ \end{cases}$$

Solution 5.3.1-2:
The trigonometric Fourier series coefficients are defined for $k \in \mathbb{Z}^+$ as:

$$A_k = \frac{2}{T}\int_{-T/2}^{+T/2} \cos(\omega_k t)\,\mathrm{d}t$$
$$B_k = \frac{2}{T}\int_{-T/2}^{+T/2} \sin(\omega_k t)\,\mathrm{d}t$$

with $\omega_k = 2k\pi/T$.
Because of the fact that an integral of an odd function multiplied with an even function over a symmetrical range equals zero, we must conclude:

$$B_k = 0$$

In addition, it is easy to see that $A_0 = 0$. Therefore, we can focus on A_k for $k \in \mathbb{Z}_0^+$.
Given the symmetry of $x(t)$ it is easy to reduce A_k to:

$$\begin{aligned}
A_k &= 2\frac{2}{T}\int_0^{T/2} x(t)\cos(\omega_k t) dxt \\
&\downarrow x(t) = -4t + 2, \qquad t \in [0:1], \text{ and } T = 2 \text{ and } \omega_k = k\pi \\
&= 2\int_0^1 (-4t+2)\cos(k\pi t)\,\mathrm{d}t \\
&= -8\int_0^1 t\cos(k\pi t)\,\mathrm{d}t + 4\int_0^1 \cos(k\pi t)\,\mathrm{d}t \\
&\downarrow \text{prepare for partial integration (1st term) and integrate (2nd term)} \\
&= -8\frac{1}{k\pi}\int_{t=0}^{t=1} t\,\mathrm{d}(\sin(k\pi t)) + 4\frac{1}{k\pi}\left[\sin(k\pi t)\right]_{t=0}^{t=1} \\
&\downarrow \text{partial integration (1st term)} \\
&= \frac{-8}{k\pi}\left(\left[t\sin(k\pi t)\right]_{t=0}^{t=1} - int_0^1 \sin(k\pi t)\,\mathrm{d}t\right) + \frac{4}{k\pi}\left(\sin(k\pi) - 0\right) \\
&= \frac{-8}{k\pi}\left(\sin(k\pi) - 0 - \frac{1}{k\pi}\left[\cos(k\pi t)\right]_{t=0}^{t=1}\right) + \frac{4}{k\pi}\sin(k\pi) \\
&= \underbrace{\frac{-4}{k\pi}\sin(k\pi)}_{=0, k\in\mathbb{Z}_0^+} + \frac{8}{(k\pi)^2}\left(\cos(k\pi) - 1\right) \\
&= \begin{cases} \frac{-16}{(k\pi)^2}, & k \in 2\mathbb{Z}^+ + 1 \\ 0, & k \in 2\mathbb{Z}_0^+ \end{cases}
\end{aligned}$$

This leads to the following conclusion:

$$A_k = \begin{cases} \frac{-16}{(k\pi)^2}, & k \in 2\mathbb{Z}^+ + 1 \\ 0, & k \in 2\mathbb{Z}^+ \end{cases}$$
$$B_k = 0, \qquad k \in \mathbb{Z}^+$$

Solution 5.3.3-1:
In the interval $[0, 2]$, $x(t)$ can be written as:

$$x(t) = \frac{3\pi}{2}(t-1)$$

Therefore:

$$X_k = \frac{1}{2}\int_0^2 \frac{3\pi}{2}(t-1)\,\mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t$$

Let's start by treating the special case with $k = 0$:

$$\begin{aligned} C_0 &= \frac{1}{2}\int_0^2 \frac{3\pi}{2}(t-1)\,\mathrm{d}t \\ &= \frac{3\pi}{4}\left[\frac{t^2}{2} - t\right]_0^2 \\ &= \frac{3\pi}{4}(2-2-0+0) = 0 \end{aligned}$$

This is not surprising as we can see from the original graph that the DC value of the waveform equals zero.
Let's continue with the more general case with $k \neq 0$:

$$\begin{aligned} X_k &= \frac{3\pi}{4}\int_0^2 (t-1)\,\mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t \\ &\downarrow \text{prepare for partial integration} \\ &= \frac{3\pi}{4}\,\frac{1}{-\mathrm{j}k\pi}\int_{t=0}^{t=2}(t-1)\,\mathrm{d}\,\mathrm{e}^{-\mathrm{j}k\pi t} \\ &\downarrow \text{partial integration} \\ &= \frac{3j}{4k}\left(\left[(t-1)\,\mathrm{e}^{-\mathrm{j}k\pi t}\right]_0^2 - \int_0^2 \mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t\right) \\ &= \frac{3j}{4k}\left(\left[(t-1)\,\mathrm{e}^{-\mathrm{j}k\pi t}\right]_0^2 - \frac{1}{-\mathrm{j}k\pi}\left[\mathrm{e}^{-\mathrm{j}k\pi t}\right]_0^2\right) \\ &= \frac{3j}{4k}\left(\left(\underbrace{\mathrm{e}^{-\mathrm{j}k2\pi}}_{=1} + 1\right) - \frac{1}{-\mathrm{j}k\pi}\left(\underbrace{\mathrm{e}^{-\mathrm{j}k2\pi}}_{=1} - 1\right)\right) \\ &= \frac{3j}{2k} \end{aligned}$$

To conclude:

$$X_k = \begin{cases} 0 & k = 0, \\ \frac{3j}{2k} & k \in \mathbb{Z}_0 \end{cases}$$

Now let's use Octave/Matlab to check our result. We make use of the auxiliary functions specified in the appendices of the course notes.

```
# generate the abcissa
t = linspace( -3, 3, 100 );

# generate the Fourier coefficients that we just calculated.
Cvec(1) = 0;
Cvec(2:101) = 3.*j./2./linspace(1, 100, 100);

# plot the result
plot( t, efsRF( Cvec, 2, t, 100 ) );
```

Solution 5.3.3-2:
In the interval $[0, 2]$, $x(t)$ can be written as

$$x(t) = \begin{cases} 2 - 4t & t \in [0, 1] \\ -6 + 4t & t \in [1, 2] \end{cases}$$

Therefore:

$$X_k = \frac{1}{2}\left(\int_0^1 (2-4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t + \int_1^2 (-6+4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t\right)$$

Let's start with the special case for $k = 0$:

$$C_0 = \frac{1}{2}\left(\int_0^1 (2-4t)\,\mathrm{d}t + \int_1^2 (-6+4t)\,\mathrm{d}t\right) = 0$$

Now, let's consider the more general case if $k \neq 0$:

$$\begin{aligned}
X_k &= \frac{1}{2}\left(\int_0^1 (2-4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t + \int_1^2 (-6+4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t\right) \\
&\quad\downarrow \text{prepare for partial integration} \\
&= \frac{j}{2k\pi}\left(\int_{t=0}^{t=1} (2-4t)\,\mathrm{d}\,\mathrm{e}^{-\mathrm{j}k\pi t} + \int_{t=1}^{t=2} (-6+4t)\,\mathrm{d}\,\mathrm{e}^{-\mathrm{j}k\pi t}\right) \\
&\quad\downarrow \text{partial integration} \\
&= \frac{j}{2k\pi}\left(\left[(2-4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\right]_0^1 + 4\int_0^1 \mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t + \left[(-6+4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\right]_1^2 - 4\int_1^2 \mathrm{e}^{-\mathrm{j}k\pi t}\,\mathrm{d}t\right) \\
&= \frac{j}{2k\pi}\left(\left[(2-4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\right]_0^1 + \frac{j4}{k\pi}\left[\mathrm{e}^{-\mathrm{j}k\pi t}\right]_0^1 + \left[(-6+4t)\,\mathrm{e}^{-\mathrm{j}k\pi t}\right]_1^2 - \frac{j4}{k\pi}\left[\mathrm{e}^{-\mathrm{j}k\pi t}\right]_1^2\right) \\
&= \frac{j}{2k\pi}\left((-2\,\mathrm{e}^{-\mathrm{j}k\pi} - 2) + \frac{j4}{k\pi}(\mathrm{e}^{-\mathrm{j}k\pi} - 1) + (2\,\mathrm{e}^{-\mathrm{j}k2\pi} + 2\,\mathrm{e}^{-\mathrm{j}k\pi}) - \frac{j4}{k\pi}(\mathrm{e}^{-\mathrm{j}k2\pi} - \mathrm{e}^{-\mathrm{j}k\pi})\right) \\
&= \frac{j}{2k\pi}\left((-2\,\mathrm{e}^{-\mathrm{j}k\pi} - 2) + \frac{j4}{k\pi}(\mathrm{e}^{-\mathrm{j}k\pi} - 1) + (2 + 2\,\mathrm{e}^{-\mathrm{j}k\pi}) - \frac{j4}{k\pi}(1 - \mathrm{e}^{-\mathrm{j}k\pi})\right)
\end{aligned}$$

We can further simplify this expression for k even:

$$\begin{aligned}
C_{k,even} &= \frac{j}{2k\pi}\left((-2-2) + \frac{j4}{k\pi}(1-1) + (2+2) - \frac{j4}{k\pi}(1-1)\right) \\
&= 0
\end{aligned}$$

and for k odd:

$$\begin{aligned}
C_{k,odd} &= \frac{j}{2k\pi}\left((+2-2) + \frac{j4}{k\pi}(-1-1) + (2-2) - \frac{j4}{k\pi}(1+1)\right) \\
&= \frac{8}{(k\pi)^2}
\end{aligned}$$

Summarized:

$$X_k = \begin{cases} 0 & k \in 2\mathbb{Z} \\ \frac{8}{(k\pi)^2} & k \in 2\mathbb{Z}+1 \end{cases}$$

Now let's use OCTAVE/MATLAB to check our result

```
# generate the abcissa
t = linspace( -3, 3, 100 );

# generate the Fourier coefficients that we just calculated.
Cvec(1) = 0;
Cvec(2:51) = 8 ./ ( (linspace(1, 50, 50)*pi).^2 );
for i = 1:25
Cvec(2*i+1) = 0;
endfor

# plot the result
plot( t, efsRF( Cvec, 2, t ) );
```

Solution 5.3.3-3:

The exponential Fourier coefficients can be calculated using their definition:

$$X_k = \frac{1}{T}\left(\int_{-T/2}^{-T/4} \frac{4}{T}\left(t + \frac{T}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t + \int_{-T/4}^{0} \frac{16}{T^2} t^2\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t + \int_{T/4}^{T/2} \mathrm{e}^{-\frac{16}{T}(t-\frac{T}{4})}\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t\right)$$

Let's consider the special case for $k = 0$:

$$\begin{aligned}
X_0 &= \frac{1}{T}\left(\int_{-T/2}^{-T/4} \frac{4}{T}\left(t+\frac{T}{2}\right)\mathrm{d}t + \int_{-T/4}^{0} \frac{16}{T^2}t^2\,\mathrm{d}t + \int_{T/4}^{T/2} \mathrm{e}^{-\frac{16}{T}(t-\frac{T}{4})}\,\mathrm{d}t\right)\\
&= \frac{1}{T}\left(\frac{4}{T}\left[\frac{t^2}{2}+\frac{T}{2}t\right]_{-T/2}^{-T/4} + \frac{16}{T^2}\left[\frac{t^3}{3}\right]_{-T/4}^{0} - \frac{T}{16}\int_{t=T/4}^{t=T/2}\mathrm{d}\,\mathrm{e}^{-\frac{16}{T}(t-\frac{T}{4})}\right)\\
&= \frac{1}{T}\left(\left(\frac{T}{8}-\frac{T}{2}-\frac{T}{2}+T\right)+\left(0+\frac{T}{12}\right)-\frac{T}{16}\left[\mathrm{e}^{-\frac{16}{T}(t-\frac{T}{4})}\right]_{T/4}^{T/2}\right)\\
&= \left(\frac{1}{8}-\frac{1}{2}-\frac{1}{2}+1\right)+\left(0+\frac{1}{12}\right)-\frac{1}{16}(\mathrm{e}^{-4}-1) = 0.26969
\end{aligned}$$

Now, let's consider the more general case if $k \neq 0$:

$$X_k = \underbrace{\frac{1}{T}\int_{-T/2}^{-T/4}\frac{4}{T}\left(t+\frac{T}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t}_{Q_k} + \underbrace{\frac{1}{T}\int_{-T/4}^{0}\frac{16}{T^2}t^2\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t}_{R_k} + \underbrace{\frac{1}{T}\int_{T/4}^{T/2}\mathrm{e}^{-\frac{16}{T}(t-\frac{T}{4})}\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t}_{S_k}$$

Let's treat the terms Q_k, R_k and S_k separately to keep the calculations focused. We'll start with Q_k.

$$\begin{aligned}
Q_k &= \frac{1}{T}\int_{-T/2}^{-T/4}\frac{4}{T}\left(t+\frac{T}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t\\
&\downarrow \text{prepare for partial integration}\\
&= \frac{1}{T}\left(\frac{2\mathrm{j}}{k\pi}\int_{t=-T/2}^{t=-T/4}\left(t+\frac{T}{2}\right)\mathrm{d}\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\right)\\
&\downarrow \text{partial integration}\\
&= \frac{2\mathrm{j}}{k\pi T}\left(\left[\left(t+\frac{T}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\right]_{-T/2}^{-T/4} - \int_{-T/2}^{-T/4}\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t\right)\\
&= \frac{2\mathrm{j}}{k\pi T}\left(\left[\left(t+\frac{T}{2}\right)\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\right]_{-T/2}^{-T/4} - \frac{jT}{2k\pi}\left[\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\right]_{-T/2}^{-T/4}\right)\\
&= \frac{2\mathrm{j}}{k\pi T}\left(\left(\left(-\frac{T}{4}+\frac{T}{2}\right)\mathrm{e}^{+j\frac{k\pi}{2}} - \left(-\frac{T}{2}+\frac{T}{2}\right)\mathrm{e}^{jk\pi}\right) - \frac{jT}{2k\pi}\left(\mathrm{e}^{j\frac{k\pi}{2}} - \mathrm{e}^{jk\pi t}\right)\right)\\
&= \frac{j^{k+1}}{2k\pi} + \frac{j^k-(-1)^k}{(k\pi)^2}
\end{aligned}$$

Let's treat R_k next:

$$
\begin{aligned}
R_k &= \frac{1}{T}\int_{-T/4}^{0} \frac{16}{T^2} t^2\, \mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t \\
&\downarrow \text{prepare for partial integration} \\
&= \frac{1}{T}\frac{16}{T^2}\frac{jT}{2k\pi}\int_{t=-T/4}^{t=0} t^2\,\mathrm{d}\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}} \\
&\downarrow \text{partial integration} \\
&= \frac{8j}{k\pi T^2}\left[t^2\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\right]_{-T/4}^{0} - \frac{16j}{k\pi T^2}\int_{-T/4}^{0}\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,t\,\mathrm{d}t \\
&\downarrow \text{prepare for partial integration} \\
&= -\frac{8j}{k\pi T^2} - \frac{T^2}{16}\,\mathrm{e}^{j\frac{k\pi}{2}} - \frac{16j}{k\pi T^2}\frac{jT}{2k\pi}\int_{-T/4}^{0}\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,t\,\mathrm{d}t \\
&\downarrow \text{partial integration} \\
&= -\frac{j}{2k\pi}\,\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}} + \frac{8}{(k\pi)^2T}\left[t\,\mathrm{e}^{-\mathrm{j}\frac{k2\pi}{T}t}\right]_{-T/4}^{0} - \frac{8}{(k\pi)^2T}\int_{-T/4}^{0}\mathrm{e}^{-\mathrm{j}\frac{k2\pi}{T}t}dxt \\
&= -\frac{j}{2k\pi}\,\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}} + \frac{2}{(k\pi)^2}\,\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}} - \frac{8}{(k\pi)^2T}\frac{jT}{2k\pi}\left[\mathrm{e}^{-\mathrm{j}\frac{k2\pi}{T}t}\right]_{-T/4}^{0} \\
&= -\frac{j}{2k\pi}\,\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}} + \frac{2}{(k\pi)^2}\,\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}} - \frac{4j}{(k\pi)^3}\left(1-\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}}\right) \\
&= \left(-\frac{j}{2k\pi} + \frac{2}{(k\pi)^2} + \frac{4j}{(k\pi)^3}\right)\mathrm{e}^{\mathrm{j}\frac{k\pi}{2}} - \frac{4j}{(k\pi)^3}
\end{aligned}
$$

And finally S_k:

$$
\begin{aligned}
S_k &= \frac{1}{T}\int_{T/4}^{T/2}\mathrm{e}^{-\frac{16}{T}(t-\frac{T}{4})}\,\mathrm{e}^{-\mathrm{j}\frac{2k\pi t}{T}}\,\mathrm{d}t \\
&= \frac{1}{T}\int_{T/4}^{T/2}\mathrm{e}^{-\left(\frac{16}{T}+\mathrm{j}\frac{2k\pi}{T}\right)t+\frac{T}{4}}\,\mathrm{d}t \\
&\downarrow \text{prepare for integration} \\
&= \frac{1}{T}\frac{-T}{16+j2k\pi}\int_{t=T/4}^{t=T/2}\mathrm{d}\,\mathrm{e}^{-\left(\frac{16}{T}+\mathrm{j}\frac{2k\pi}{T}\right)t+\frac{T}{4}} \\
&= \frac{-1}{16+\mathrm{j}2k\pi}\left[\mathrm{e}^{-\left(\frac{16}{T}+\mathrm{j}\frac{2k\pi}{T}\right)t+\frac{T}{4}}\right]_{T/4}^{T/2} \\
&= \frac{-1}{16+\mathrm{j}2k\pi}\left(\mathrm{e}^{-8-jk\pi+\frac{T}{4}} - \mathrm{e}^{-4-\mathrm{j}\frac{k\pi}{2}+\frac{T}{4}}\right)
\end{aligned}
$$

Now let's use OCTAVE/MATLAB to check our result

```
# generate the abcissa
T=4;
t = linspace( -T/2, T/2, 1001 );

# generate the Fourier coefficients that we just calculated.
k = linspace(1, 100, 100 );
clear Cvec;
Cvec = zeros( 1, length(k)+1 );
Cvec(1) = 0.26969;
Cvec(2:101) += j.^(k+1) ./ (2 * k * pi ) + (j.^k - (-1).^k ) ./ ( ( k*pi).^2 );
Cvec(2:101) += (-j./(2*k*pi) + 2 ./ ((k * pi).^2) ...
+ 4*j./((k*pi).^3)) .* (j.^k) - 4*j ./ ((k*pi).^3);
Cvec(2:101) += -1 ./ ( 16 + j*2*k*pi) .* ...
(exp( -4 -j*k*pi) - exp( -j * k * pi / 2 ));

# plot the result
plot( t, efsRF( Cvec, 4, t ) );
```

Solution 5.3.4-1:

Using the conversion formulae, we easily obtain:

$$\begin{aligned} X_0 &= 0 \\ X_k &= \mathrm{j}\frac{3}{2k}, \qquad k \in \mathbb{Z}_0^+ \\ X_{-k} &= -\mathrm{j}\frac{3}{2k}, \qquad k \in \mathbb{Z}_0^+ \end{aligned}$$

Or when we allow $k \in \mathbb{Z}_0$:

$$\begin{aligned} X_0 &= 0 \\ X_k &= \mathrm{j}\frac{3}{2k}, \qquad k \in \mathbb{Z}_0 \end{aligned}$$

Solution 5.3.4-2:
Using the conversion formulae, we easily obtain:

$$\begin{aligned} X_0 &= 0 \\ X_k &= \begin{cases} \frac{-8}{(k\pi)^2}, & k \in 2\mathbb{Z}^+ + 1 \\ 0, & k \in 2\mathbb{Z}_0^+ \end{cases} \\ X_{-k} &= \begin{cases} \frac{-8}{(k\pi)^2}, & k \in 2\mathbb{Z}^+ + 1 \\ 0, & k \in 2\mathbb{Z}_0^+ \end{cases} \end{aligned}$$

Or when we allow $k \in \mathbb{Z}$:

$$X_k = \begin{cases} \frac{-8}{(k\pi)^2}, & k \in 2\mathbb{Z} + 1 \\ 0, & k \in 2\mathbb{Z} \end{cases}$$

Solution 5.3.4-3:
In general:

$$\begin{aligned} A_k &= X_k + X_{-k} \\ B_k &= \mathrm{j}(X_k - X_{-k}) \end{aligned}$$

Given

$$X_k = \begin{cases} 0 & k = 0, \\ \frac{3\mathrm{j}}{2k} & k \in \mathbb{Z}_0 \end{cases}$$

this results in:

$$\begin{aligned} A_k &= 0 \\ B_k &= \mathrm{j}\left(\frac{3\mathrm{j}}{2k} - \frac{3\mathrm{j}}{-2k}\right) = -3/k, \qquad \forall k \in \mathbb{Z}_0^+ \end{aligned}$$

Solution 5.3.4-4:
In general:

$$\begin{aligned} A_k &= X_k + X_{-k} \\ B_k &= \mathrm{j}(X_k - X_{-k}) \end{aligned}$$

Given

$$X_k = \begin{cases} 0, & k \in 2\mathbb{Z} \\ \frac{8}{(k\pi)^2}, & k \in 2\mathbb{Z} + 1 \end{cases}$$

this results in:

$$\begin{aligned} A_k &= \begin{cases} 0, & k \in 2\mathbb{Z}^+ \\ \frac{16}{(k\pi)^2}, & k \in 2\mathbb{Z}^+ + 1 \end{cases} \\ B_k &= 0, \qquad \forall k \in \mathbb{Z}_0^+ \end{aligned}$$

Solution 5.3.4-5:

$$
\begin{aligned}
x(t) &\sim \sum_{k=-\infty}^{+\infty} X_k \, e^{jk\omega_1 t} \\
&\Big\downarrow \; e^{j\alpha} = \cos\alpha + j\sin\alpha \\
&\sim \sum_{k=-\infty}^{+\infty} X_k \left(\cos(k\omega_1 t) + j\sin(k\omega_1 t)\right) \\
&\sim X_0 + \sum_{k=1}^{+\infty} \left(X_k\cos(k\omega_1 t) + X_{-k}\cos(-k\omega_1 t) + jX_k\sin(k\omega_1 t) + jX_{-k}\sin(-k\omega_1 t)\right) \\
&\Big\downarrow \; \cos(-\alpha) = \cos\alpha \;\text{ and }\; \sin(-\alpha) = -\sin\alpha \\
&\sim X_0 + \sum_{k=1}^{+\infty} \left(X_k\cos(k\omega_1 t) + X_{-k}\cos(k\omega_1 t) + jX_k\sin(k\omega_1 t) - jX_{-k}\sin(k\omega_1 t)\right) \\
&\sim \underbrace{X_0}_{A_0/2} + \sum_{k=1}^{+\infty} \left(\underbrace{(X_k + X_{-k})}_{A_k} \cos(k\omega_1 t) + \underbrace{j\,(X_k - X_{-k})}_{B_k} \sin(k\omega_1 t) \right)
\end{aligned}
$$

Solution 5.4.3-1:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2}} \, e^{-j\omega t} \, dt \\
&= \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2} - j\omega t} \, dt \\
&\Big\downarrow \; \text{complete the square} \\
&= \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2} - j\omega t + \frac{\omega^2\sigma^2}{2}} \, e^{-\frac{\omega^2\sigma^2}{2}} \, dt \\
&= e^{-\frac{\omega^2\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{t}{\sigma\sqrt{2}} + j\frac{\sigma\sqrt{2}}{2}\omega\right)^2} \, dt \\
&\Big\downarrow \; \text{substitute } u = \frac{t}{\sigma\sqrt{2}} + j\frac{\sigma\sqrt{2}}{2}\omega \\
&= e^{-\frac{\omega^2\sigma^2}{2}} \sigma\sqrt{2} \underbrace{\int_{-\infty}^{+\infty} e^{-u^2} \, du}_{=\sqrt{\pi}} \\
&= \sigma\sqrt{2\pi} \, e^{-\frac{\omega^2\sigma^2}{2}}
\end{aligned}
$$

Solution 5.4.3-2:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{+\infty} u(t) \, e^{-\alpha t} \, e^{-j\omega t} \, dt \\
&= \int_{0}^{+\infty} e^{-(\alpha + j\omega)t} \, dt \\
&= \frac{-1}{\alpha + j\omega} \left[e^{-(\alpha+j\omega)t} \right]_{t=0}^{t=+\infty} \\
&= \frac{-1}{\alpha + j\omega} (0 - 1) = \frac{1}{\alpha + j\omega}
\end{aligned}
$$

Solution 5.4.3-3:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{+\infty} \mathrm{u}(t)\, t\, \mathrm{e}^{-\alpha t}\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t \\
&= \int_{0}^{+\infty} t\, \mathrm{e}^{-(\alpha+\mathrm{j}\omega)t}\, \mathrm{d}t \\
&\downarrow \text{prepare for partial integration} \\
&= \frac{-1}{\alpha+\mathrm{j}\omega} \int_{t=0}^{t=+\infty} t\, \mathrm{d}\, \mathrm{e}^{-(\alpha+\mathrm{j}\omega)t} \\
&\downarrow \text{partial integration} \\
&= \frac{-1}{\alpha+\mathrm{j}\omega} \left(\left[t\, \mathrm{e}^{-(\alpha+\mathrm{j}\omega)t} \right]_{t=0}^{t=+\infty} - \int_{0}^{+\infty} \mathrm{e}^{-(\alpha+\mathrm{j}\omega)t}\, \mathrm{d}t \right) \\
&= \frac{-1}{\alpha+\mathrm{j}\omega} \left((0-0) - \frac{-1}{\alpha+\mathrm{j}\omega} \left[\mathrm{e}^{-(\alpha+\mathrm{j}\omega)t} \right]_{t=0}^{t=+\infty} \right) \\
&= \frac{-1}{(\alpha+\mathrm{j}\omega)^2} (0-1) \\
&= \frac{1}{(\alpha+\mathrm{j}\omega)^2}
\end{aligned}
$$

Solution 5.4.3-4:
As we know that the signal $x(t)$ is periodic, we can calculate its exponential Fourier series coefficients. These are:

$$
X_k = \begin{cases} 0 & k = 0, \\ \frac{3\mathrm{j}}{2k} & k \in \mathbb{Z}_0 \end{cases}
$$

In addition we have an equation that allows us converting these harmonic numbers into a Fourier transform:

$$
X(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} X_k\, \delta(\omega - \omega_k)
$$

We therefore obtain:

$$
X(\omega) = 2\pi \sum_{k \in \mathbb{Z}_0} \frac{3\mathrm{j}}{2k}\, \delta(\omega - \pi k)
$$

Solution 5.4.3-5:
As we know that the signal $x(t)$ is periodic, we can calculate its exponential Fourier series coefficients. These are:

$$
X_k = \begin{cases} 0 & k \in 2\mathbb{Z} \\ \frac{8}{(k\pi)^2} & k \in 2\mathbb{Z}+1 \end{cases}
$$

In addition we have an equation that allows us converting these harmonic numbers into a Fourier transform:

$$
X(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} X_k\, \delta(\omega - \omega_k)
$$

We therefore obtain:

$$
X(\omega) = 2\pi \sum_{k \in 2\mathbb{Z}+1} \frac{8}{(k\pi)^2}\, \delta(\omega - \pi k)
$$

Solution 5.4.3-6:

$$
F(\omega) = \frac{2\mathrm{j}}{\omega}(2\,\mathrm{e}^{-\mathrm{j}2\omega} + \mathrm{e}^{-\mathrm{j}\omega})
$$

or after clever reworking:

$$
F(\omega) = \frac{2}{\omega}(\sin(\omega) - \sin(2\omega)) + \frac{\mathrm{j}2}{\omega}(\cos(\omega) - \cos(2\omega))
$$

Solution 5.4.3-7:
$F(\omega) = \frac{2A}{j\omega} \cdot (1 - \cos\left(\omega \frac{T}{2}\right))$

Solution 5.4.3-8:
$F(\omega) = \frac{2}{\omega}(\sin(2\omega) - \sin(\omega))$

Solution 5.4.3-9:
$F(\omega) = \frac{A}{B}[\frac{-B}{j\omega}\,e^{-j\omega B} + \frac{1}{\omega^2}\,e^{-j\omega B} - \frac{1}{\omega^2}]$

Solution 5.4.3-10:
$F(\omega) = 10\pi j\,[\delta(\omega + 50) - \delta(\omega - 50)]$

Solution 5.4.3-11:
$F(\omega) = \frac{A}{a + j\omega}$

Solution 5.4.3-12:
$F(\omega) = \frac{2}{j\omega}$

Solution 5.4.3-13:
This integral does not converge. Add an extra convergence factor e^{-at} with $a \in \mathbb{R}_0^+$ and afterwards calculate the limiting case for $a \to 0$. This leads to: $F(\omega) = \frac{j\omega}{\omega_0^2 - \omega^2}$

Solution 5.4.3-14:
The Fourier integral does not converge. There are two options:

1. Consider the transform of $u(t)\,\cos\omega_0 t$ and calculate the limit for $\omega_0 \to 0$.
2. Consider $u(t) = \frac{1+\mathrm{sgn}(t)}{2}$ and use the transform of $\mathrm{sgn}(t)$ we determined earlier.

Note that considering $A\,e^{-at}\,u(t) \xrightarrow{\mathcal{F}} \frac{A}{j\omega + a}$ and letting $a \to 0$ does not yield the correct answer. The reason for this is that the classical Fourier theory with functions is not consistent. One needs a Fourier theory with distributions to make it consistent. We don't go that far. However, the idea of working with convergence factors is very useful as we will see when treating the Laplace transform.

Solution 5.4.4-1:
The time-frequency symmetry property tells us to interchange the rol of time-domain function and frequency domain function, invert the frequency axis and multiply the frequency domain by 2π. Therefore:

$$1 \xrightarrow{\mathcal{F}} 2\pi\,\delta(-\omega)$$

which can be simplified to:

$$1 \xrightarrow{\mathcal{F}} 2\pi\,\delta(\omega)$$

Solution 5.4.4-2:
We know that:

$$\cos(\omega_0 t) = \frac{1}{\omega_0}\frac{d(\sin(\omega_0 t))}{dt}$$

Time differentiation corresponds to multiplication with $j\omega$ in the frequency domain. Therefore:

$$\begin{aligned}\cos(\omega_0 t) &\xrightarrow{\mathcal{F}} \frac{1}{\omega_0} j\omega (j\pi\,(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)) \\ &\xrightarrow{\mathcal{F}} \frac{j\omega}{\omega_0} j\pi\,\delta(\omega + \omega_0) - \frac{j\omega}{\omega_0} j\pi\,\delta(\omega - \omega_0)\end{aligned}$$

The right-hand side of this relationship is only nonzero for $\omega = \pm\omega_0$, more specifically, the first term is nonzero for $\omega = -\omega_0$, the second term for $\omega = \omega_0$. This yields:

$$\begin{aligned}\cos(\omega_0 t) &\xrightarrow{\mathcal{F}} \frac{-\mathrm{j}\omega_0}{\omega_0}\mathrm{j}\pi\,\delta(\omega+\omega_0) - \frac{\mathrm{j}\omega_0}{\omega_0}\mathrm{j}\pi\,\delta(\omega-\omega_0)\\ &\xrightarrow{\mathcal{F}} \pi\,\delta(\omega+\omega_0) + \pi\,\delta(\omega-\omega_0)\\ &\xrightarrow{\mathcal{F}} \pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right)\end{aligned}$$

Solution 5.4.4-3:
Multiplying the time scale with a factor k corresponds to dividing the frequency scale by k and reducing the magnitude by a factor of $|k|$.
Therefore:

$$\cos\left(\frac{\omega_0}{10}t\right) \xrightarrow{\mathcal{F}} 10\pi\left(\delta(10\omega-\omega_0)+\delta(10\omega+\omega_0)\right)$$

Solution 5.4.4-4:
A time delay over an amount t_0 corresponds to multiplication in the frequency domain with an exponential of $\mathrm{e}^{-\mathrm{j}\omega t_0}$. Therefore:

$$\mathrm{e}^{-\alpha(t-t_0)}\,\mathrm{u}(t-t_0) \xrightarrow{\mathcal{F}} \frac{\mathrm{e}^{-\mathrm{j}\omega t_0}}{\mathrm{j}\omega+\alpha}$$

Solution 5.4.4-5:
We can write the Dirac impulse as:

$$\delta(t) = \frac{1}{2}\frac{\mathrm{d}\,\mathrm{sgn}(t)}{\mathrm{d}t}$$

Time derivation corresponds to multiplication in the frequency domain with a factor of $\mathrm{j}\omega$. Knowing that

$$\mathrm{sgn}(t) \xrightarrow{\mathcal{F}} \frac{2}{\mathrm{j}\omega}$$

this leads to

$$\delta(t) \xrightarrow{\mathcal{F}} \mathrm{j}\omega\frac{1}{2}\frac{2}{\mathrm{j}\omega} = 1$$

Solution 5.5.3-1:
(a) $H(\omega) = \frac{1}{1+j\omega RC}$ (b) $H(\omega) = \frac{j\omega L}{R+j\omega L}$

Solution 5.5.3-2:
(a) $H(\omega) = \frac{j\omega RC}{1+j\omega RC}$ (b) $H(\omega) = \frac{1}{-\omega^2 LC+j\omega RC+1}$

Solution 5.5.3-3:
(a) $h(t) = \delta(t) - \frac{1}{RC}\,\mathrm{e}^{-t/RC}\,\mathrm{u}(t)$ (b) $h(t) = \frac{R}{L}\,\mathrm{e}^{-tR/L}\,\mathrm{u}(t)$

Solution 5.5.3-4:
$H(\omega) = -\frac{\frac{1}{jC_2\omega}+R_2}{\frac{R_1}{1+jC_1R_1\omega}+R_3}$

Solution 5.5.3-5:
$H(\omega) = \frac{jC_1R_1(R_2+R_3)\omega}{R_2(1+jC_1R_1\omega)} \cdot \frac{1}{1+jC_2R_4\omega}$

Solution 5.5.3-6:
$H(\omega) = -\frac{R_4}{R_3} \cdot \frac{1+jCR_2\omega}{1+jC(R_1+R_2)\omega}$

Solution 5.5.3-7:
$H(\omega) = -\frac{R_4}{R_3} \cdot \frac{1+jCR_2\omega}{1+jC(R_1+R_2)\omega} \cdot \frac{jLR_6\omega+R_5R_6}{R_5R_6+jL(R_5+R_6)\omega}$

Solution 5.5.3-8:
$H(\omega) = \frac{j\omega-\frac{R_2}{R_1R_3C}}{j\omega+\frac{1}{R_3C}}$

Chapter 6

The Laplace transform

Solution 6.4-5:
$f(0) = 2 \qquad \text{and} \qquad f(\infty) = 2$

Solution 6.4-6:
$v(0) = 1 \qquad \text{and} \qquad v(\infty) = 0$

Solution 6.4-7:
$v(0) = 0 \qquad \text{and} \qquad v(\infty) = 10$

Solution 6.4-8:
$f(0) = -2 \qquad \text{and} \qquad f(\infty) = 0$

Solution 6.7.3-1:
$R = 15\,\Omega, L = 0.2\,\text{H}, h(t) = 75\,\text{e}^{-100t}\,\text{u}(t)$, stable

Solution 6.7.3-2:
$$H(s) = \frac{20}{(s+2)^2}, \text{ stable}$$

Solution 6.7.3-3:
$$H(s) = \frac{0.03s^2 + 13121.6s + 102400}{(s-8)(s+320)}, \text{ unstable}$$

Solution 6.7.3-4:
$$H(s) = \frac{25s}{(s+5)(s+20)}, \text{ stable}$$

Solution 6.9.1-1:
We omitted the factors $u(t)$.

a) $y(t) = 4 - 2\,e^{-t}$

b) $y(t) = \frac{1}{2}(t^2 + 2)\,e^{-t}$

c) $y(t) = (2 - \cos t)\,e^{-t}$

d) $y(t) = \frac{1}{10}\left[\,e^{-t}(5 - 4\cos t - 2\sin t) - e^{-3t}\right]$

e) $y(t) = \frac{2}{3}\,e^{t} - e^{-t} + \frac{4}{3}\,e^{-2t}$

f) $y(t) = \frac{1}{25}\left(12\,e^{4t} + 13\,e^{-t} - 10t\,e^{-t}\right)$

g) $y(t) = e^{-2t}\cos t$

h) $y(t) = 4\,e^{2t} - e^{t}$

Solution 6.9.1-2:
We omitted the factors $u(t)$.

a) $y = e^{2t}$

b) $y = 5J_0(t)$

c) $y = t\,e^{4t}$

d) $y = \sin 4t$

d) $y = 3 - \sin t$

e) $y = t^2\,e^{-t}$

f) $y = 3 + 2t + t^2$

Solution 6.9.2-1:
$$\frac{-R_3}{Cs(R_1R_2 + R_1R_3 + R_2R_3) + R_1}$$

Solution 6.9.2-2:
$$\frac{-s}{C_2R_2}/(s^2 + \frac{s}{C_1R_1} + \frac{1}{R_1R_2C_1C_2})$$

Solution 6.9.2-3:
$R = 5\,\mathrm{k}\Omega, C = 0.5\,\mu\mathrm{F}, L = 1\,\mathrm{H}$ or $R = 5\,\mathrm{k}\Omega, C = 0.2\,\mu\mathrm{F}, L = 2.5\,\mathrm{H}$

Chapter 7

Analyzing LTI systems — time domain

Solution 7.6.3-1:
$i(t) = -3 - 3\,\mathrm{e}^{-0.8t}$

Solution 7.6.3-2:
$v_o(t) = 4 + 2\,\mathrm{e}^{-t/2}$

Solution 7.6.3-3:
$R = 6\,\Omega \qquad \text{and} \qquad C = 0.25\,\mathrm{F}$

Solution 7.6.3-4:
$h(t) = 1.25 \times 10^7\,(\,\mathrm{e}^{-5\,000t} - \mathrm{e}^{-25\,000t})\,\mathrm{u}(t)$

Solution 7.6.3-5:
$h(t) = 10323(\,\mathrm{e}^{-10\,000t} - \mathrm{e}^{-320\,000t})\,\mathrm{u}(t)$

Solution 7.6.3-6:
$h(t) = 10323\,\mathrm{e}^{-320\,000t} - 322.6\,\mathrm{e}^{-10\,000t}\,\mathrm{u}(t)$

Solution 7.6.3-7:
$h(t) = \sigma(t) + (322.6\,\mathrm{e}^{-10\,000t} - 330323\,\mathrm{e}^{-320\,000t})\,\mathrm{u}(t)$

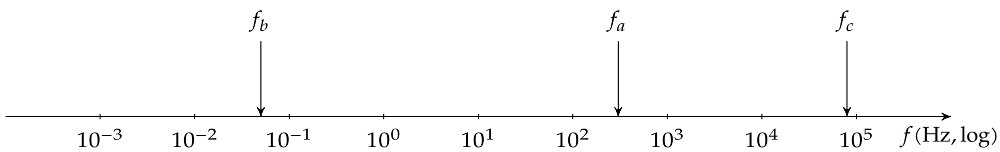
Chapter 8

Analyzing LTI systems — frequency domain

Solution 8.4.4-2:
The reasoning to calculate the required frequencies has been summarized below:

$$f_a = -2.5\,\text{dec} = 10^{-2.5} = 3.2\,\text{mHz}$$
$$f_b = 0.3\,\text{dec} = 10^{0.3} = 1.96\,\text{Hz}$$
$$f_b = 2.9\,\text{dec} = 10^{2.9} = 794\,\text{Hz}$$

Solution 8.4.4-3:
The frequency values have been indicated below:



Solution 8.6.2-1:
The values can be calculated by setting $s = j\omega$ and calculating the appropriate limiting cases.

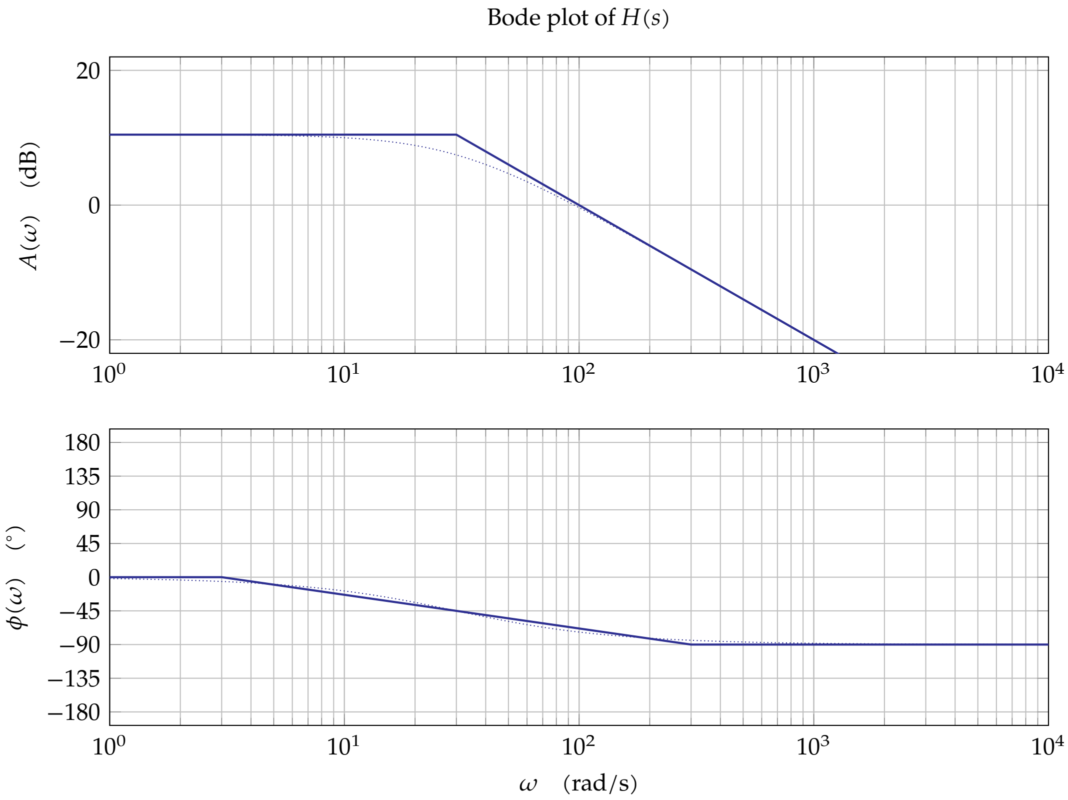
Low frequency:

$$\lim_{\omega\to 0} F(j\omega) = \lim_{\omega\to 0} \frac{-5}{+10} = \frac{1}{2}\angle -180^\circ$$
$$\lim_{\omega\to 0} G(j\omega) = \lim_{\omega\to 0} \frac{20}{100} = \frac{1}{5}\angle 0^\circ$$
$$\lim_{\omega\to 0} H(j\omega) = \lim_{\omega\to 0} \frac{-20j\omega}{100(j\omega)^3} = 0\angle -180^\circ$$

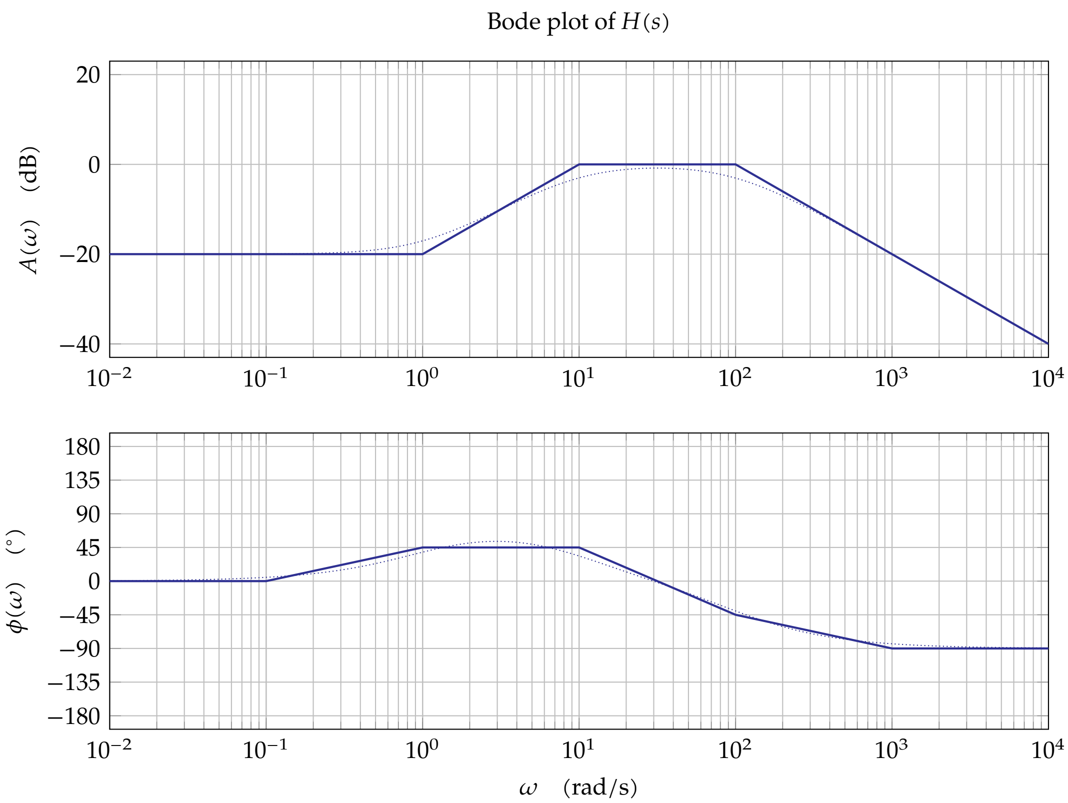
High frequency:

$$\lim_{\omega\to +\infty} F(j\omega) = \lim_{\omega\to +\infty} \frac{10(j\omega)^2}{(j\omega)^2} = 10\angle 0^\circ$$
$$\lim_{\omega\to +\infty} G(j\omega) = \lim_{\omega\to +\infty} \frac{(j\omega)^3}{(j\omega)^2} = +\infty\angle 90^\circ$$
$$\lim_{\omega\to +\infty} H(j\omega) = \lim_{\omega\to +\infty} \frac{3(j\omega)^2}{(j\omega)^5} = 0\angle -270^\circ$$

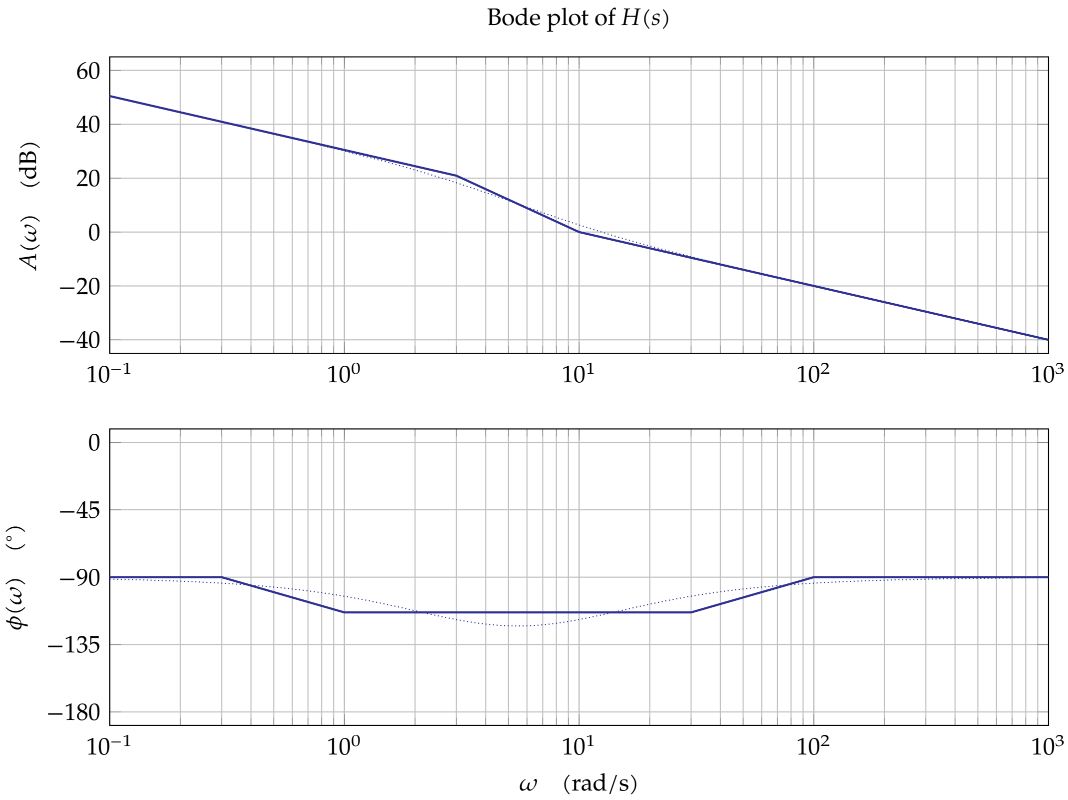
Solution 8.6.2-2:



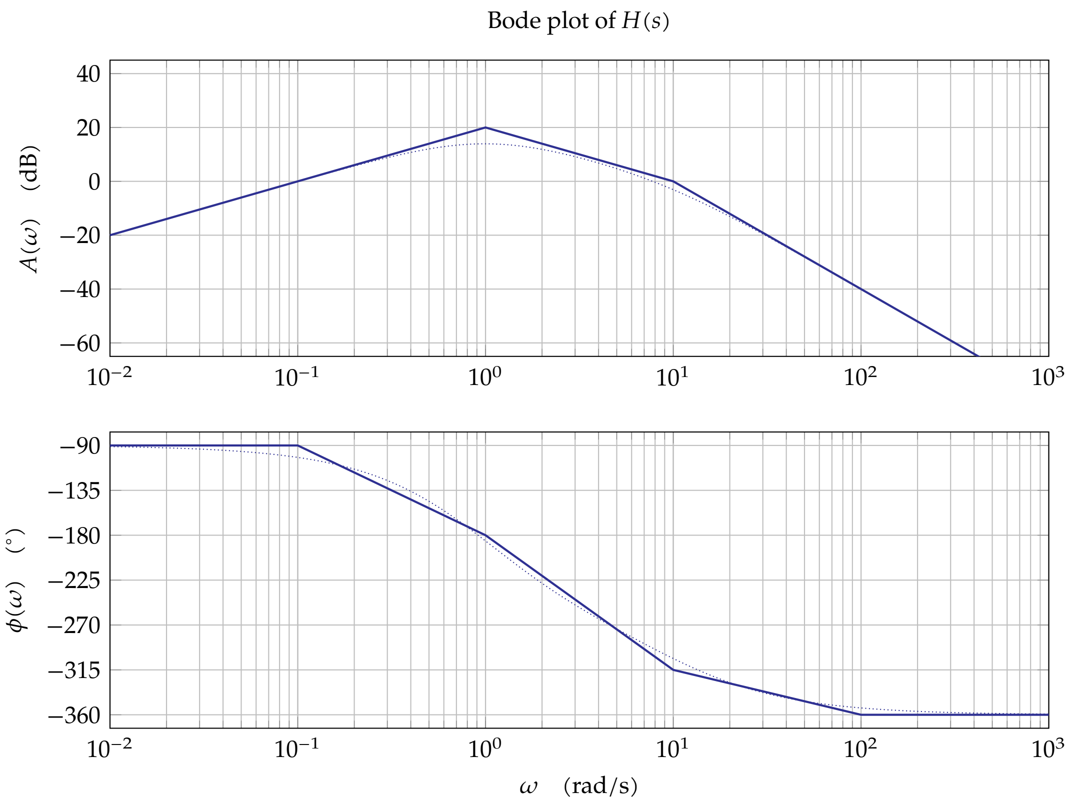
Solution 8.6.2-3:



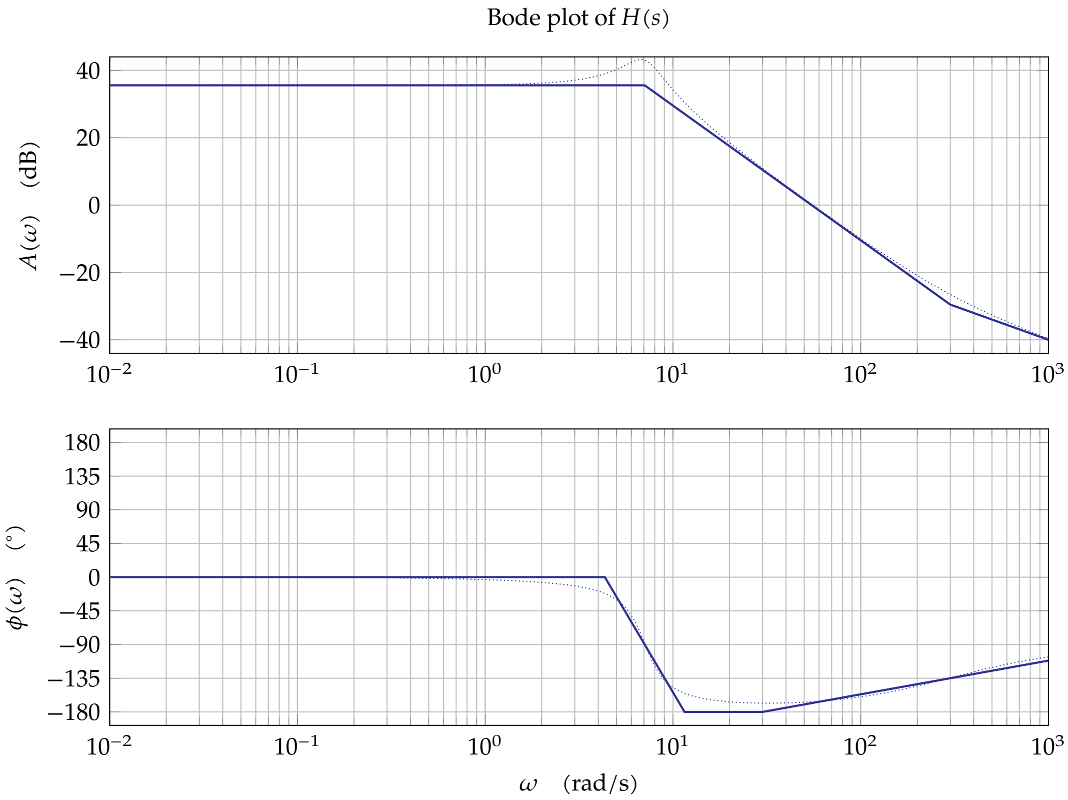
Solution 8.6.2-4:



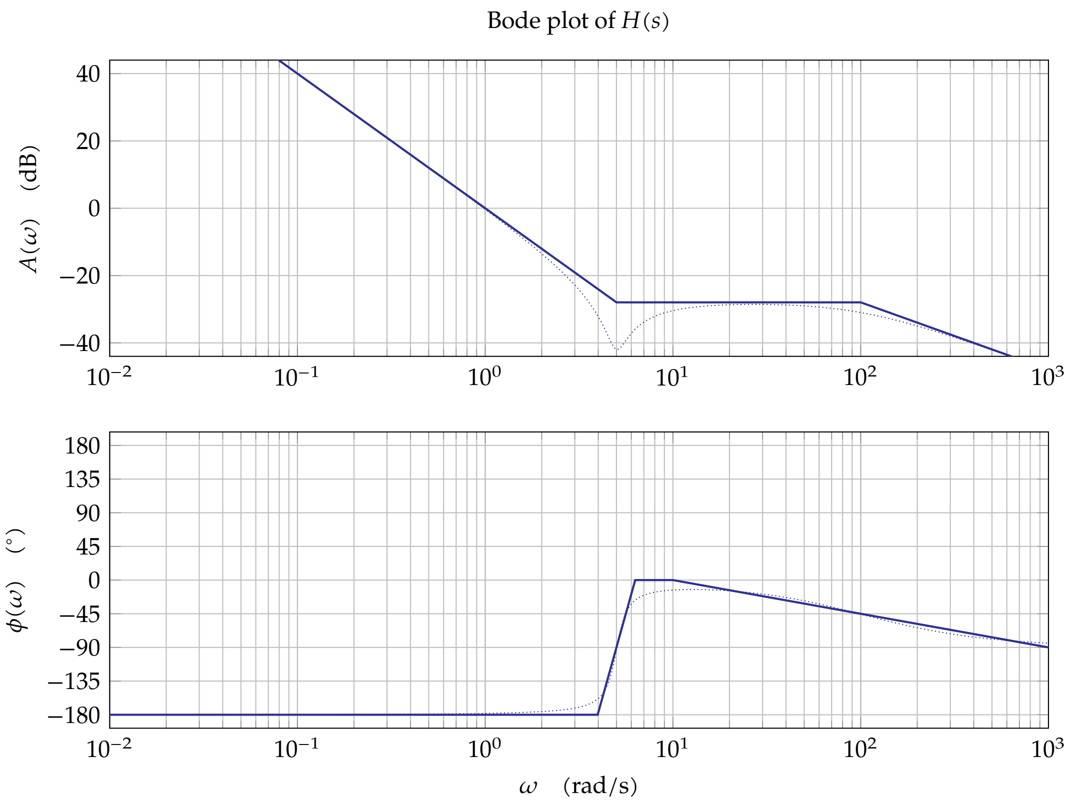
Solution 8.6.2-5:



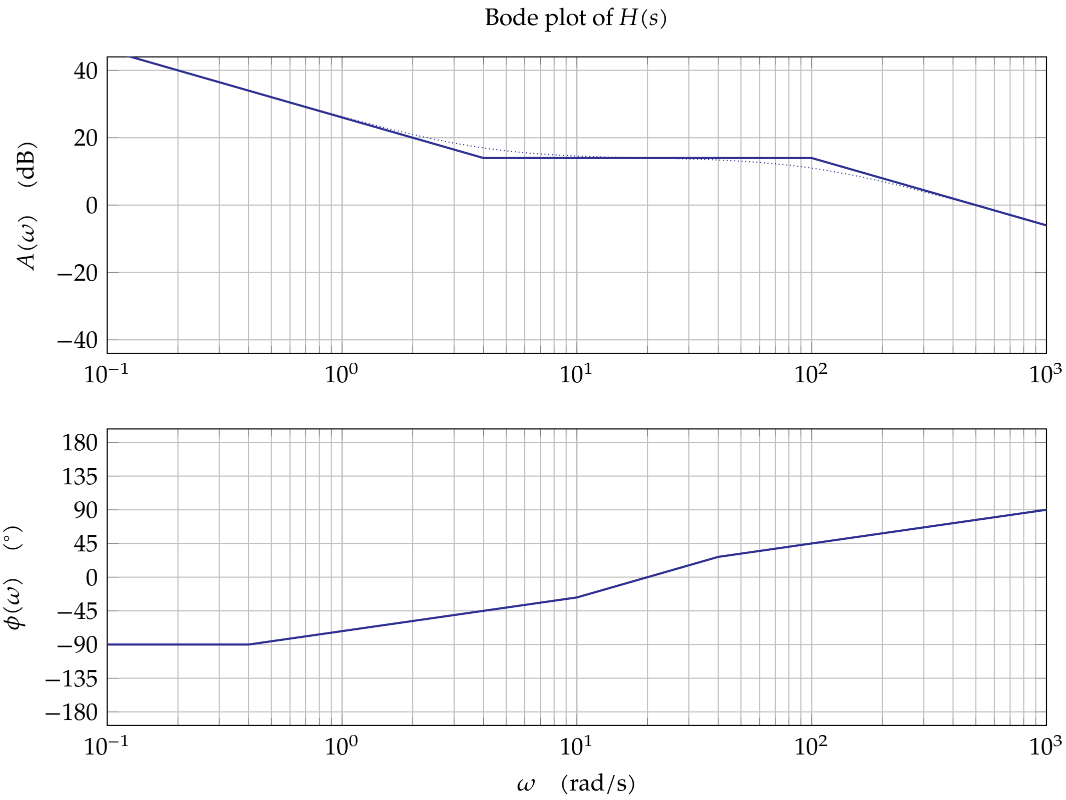
Solution 8.6.2-6:



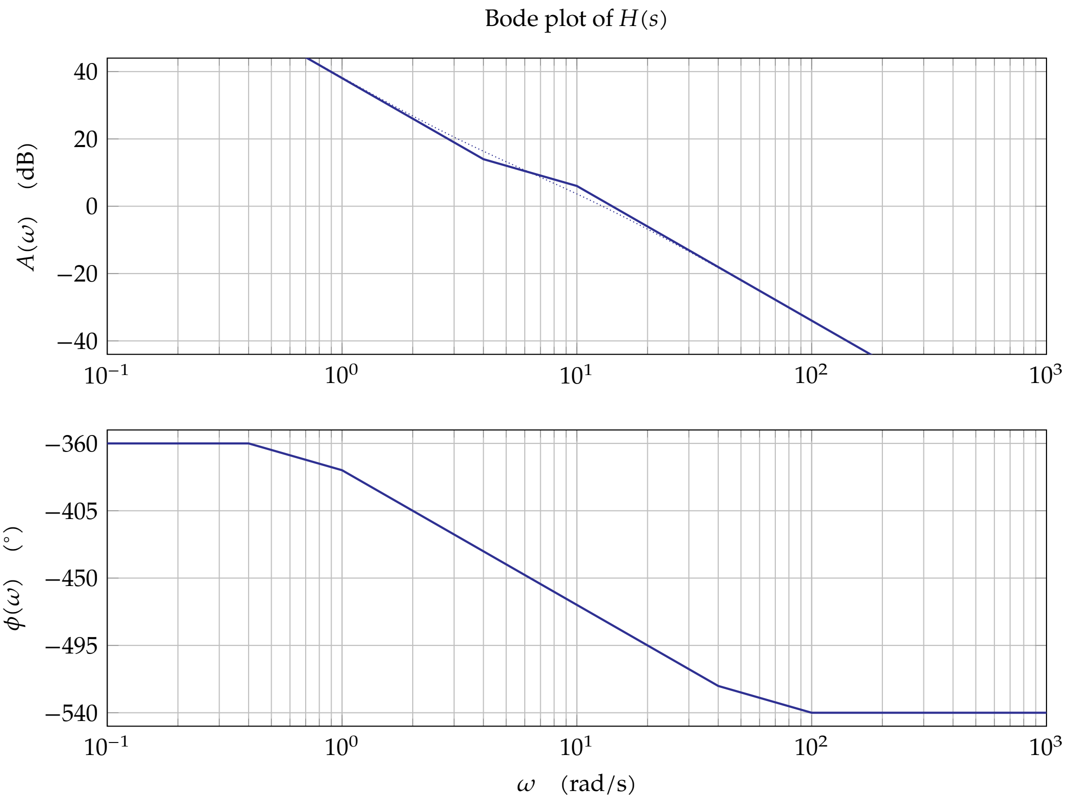
Solution 8.6.2-7:



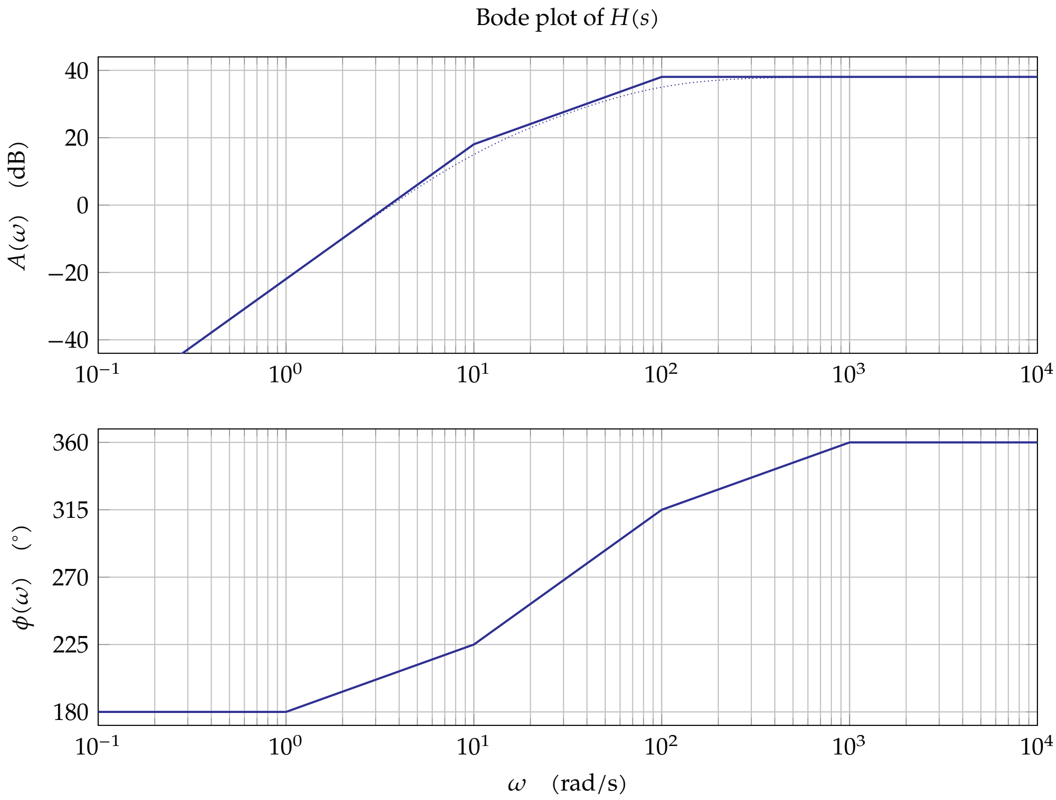
Solution 8.6.2-8:



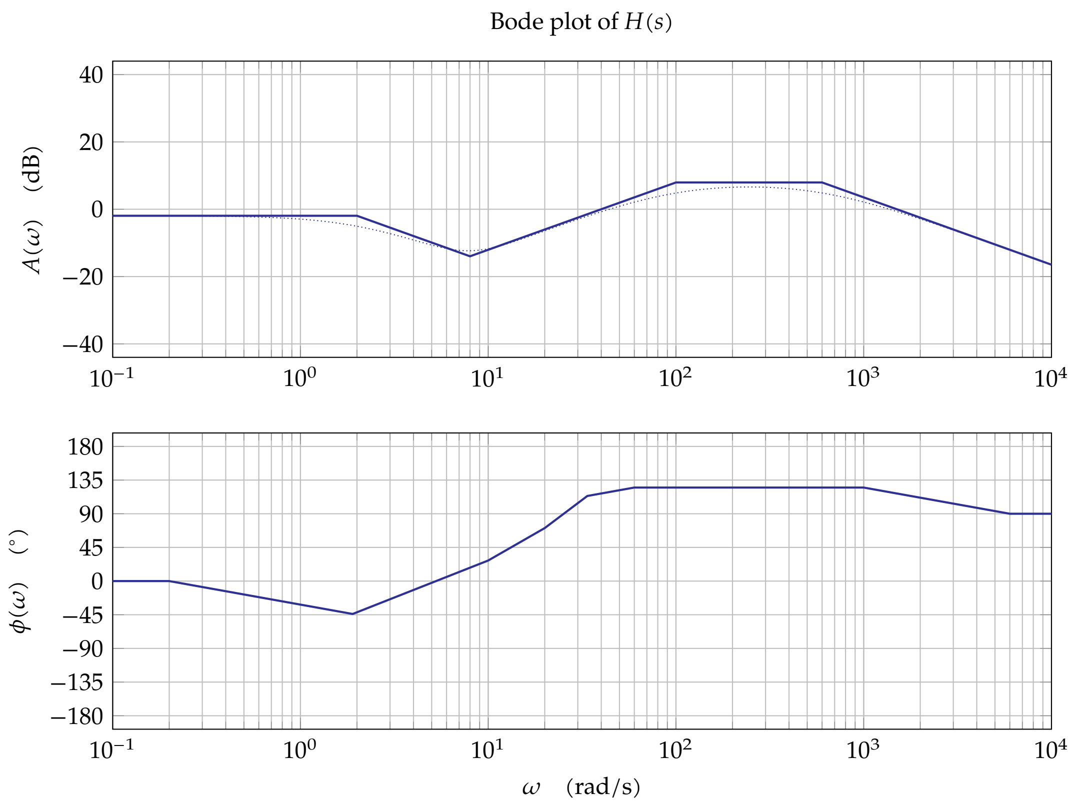
Solution 8.6.2-9:



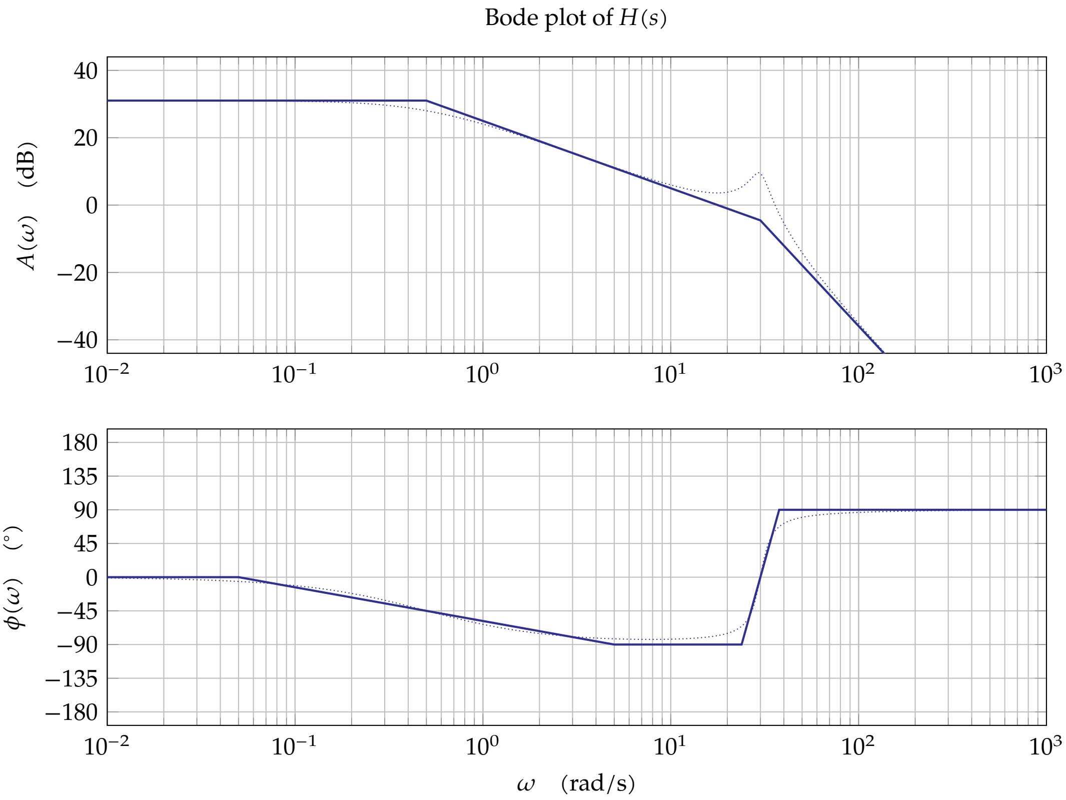
Solution 8.6.2-10:



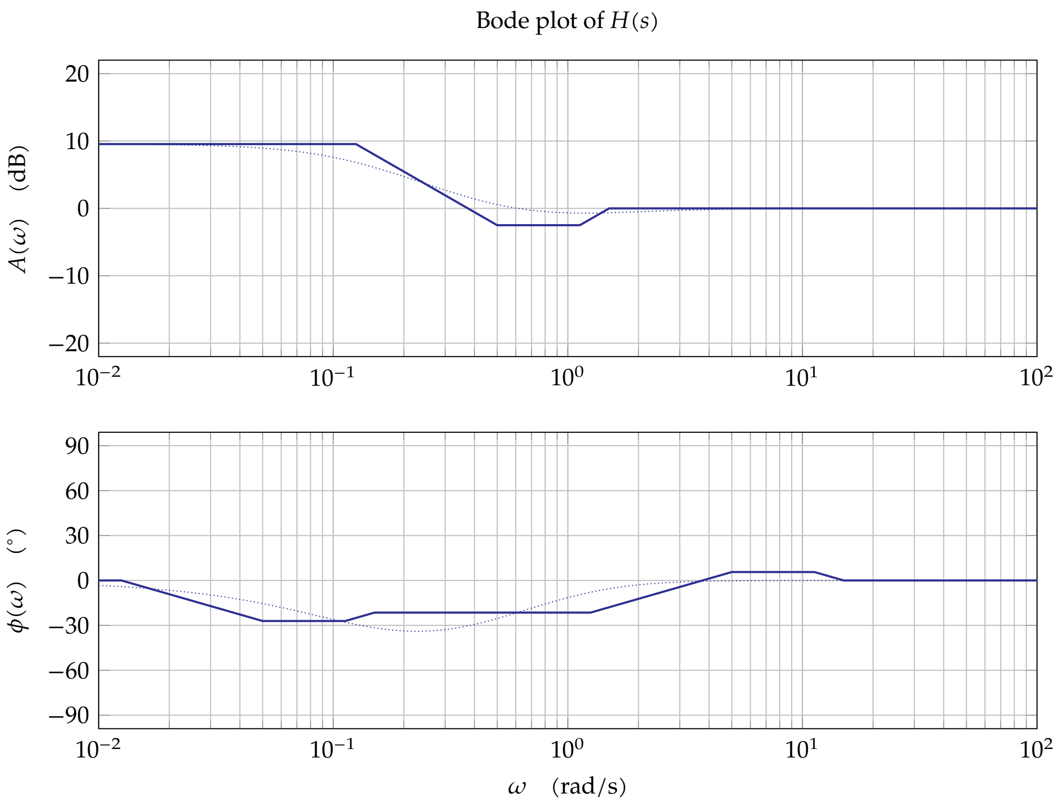
Solution 8.6.2-11:



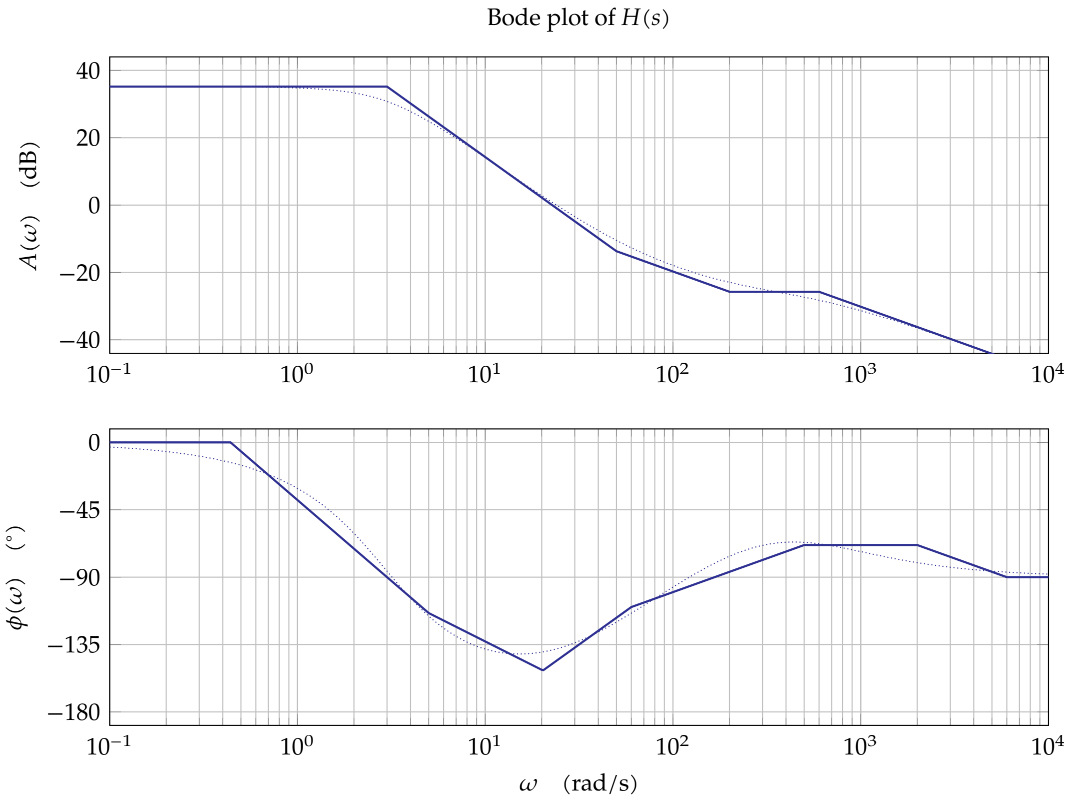
Solution 8.6.2-12:



Solution 8.6.2-13:



Solution 8.6.2-14:



Solution 8.6.2-15: